

# On Unbounded Composition Operators in $L^2$ -Spaces

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**ABSTRACT.** Fundamental properties of unbounded composition operators in  $L^2$ -spaces are studied. Characterizations of normal and quasinormal composition operators are provided. Formally normal composition operators are shown to be normal. Composition operators generating Stieltjes moment sequences are completely characterized. The unbounded counterparts of the celebrated Lambert's characterizations of subnormality of bounded composition operators are shown to be false. Various illustrative examples are supplied.

## 1. Introduction

Composition operators (in  $L^2$ -spaces over  $\sigma$ -finite spaces), which play an essential role in Ergodic Theory, turn out to be interesting objects of Operator Theory. The questions of boundedness, normality, quasinormality, subnormality, hyponormality etc. of such operators have been answered (cf. [17, 42, 34, 56, 23, 44, 30, 31, 32, 16, 19, 20, 43, 54, 10, 8, 9]; see also [18, 33, 45, 15, 47] for particular classes of composition operators). This means that the theory of bounded composition operators on  $L^2$ -spaces is well-developed.

The literature on unbounded composition operators in  $L^2$ -spaces is meagre. So far, only the questions of seminormality,  $k$ -expansivity and complete hyperexpansivity have been studied (cf. [11, 24]). Very little is known about other properties of unbounded composition operators. To the best of our knowledge, there is no paper concerning the issue of subnormality of such operators. It is a difficult question mainly because Lambert's criterion for subnormality of bounded operators (cf. [29]) is no longer valid for unbounded ones. In the present paper we show that the unbounded counterparts of the celebrated Lambert's characterizations of subnormality of bounded composition operators given in [31] fail to hold. This is achieved by proving that a composition operator satisfies the requirements of Lambert's

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characterizations if and only if it generates Stieltjes moment sequences (see Definition 2.3 and Theorem 10.4). Thus, knowing that there exists a non-subnormal composition operator which generates Stieltjes moment sequences (see [25, Theorem 4.3.3]), we obtain the above-mentioned result (see Conclusion 10.5). We point out that there exists a non-subnormal formally normal operator which generates Stieltjes moment sequences (for details see [7, Section 3.2]). This is never the case for composition operators because, as shown in Theorem 9.4, each formally normal composition operator is normal, and as such subnormal. We refer the reader to [48, 49, 50, 51] for the foundations of the theory of unbounded subnormal operators (for the bounded case see [21, 14]).

The above discussion makes plain the importance of the question of when  $C^\infty$ -vectors of a composition operator form a dense subset of the underlying  $L^2$ -space. This and related topics are studied in Section 4. In Section 3, we collect some necessary facts on composition operators. Illustrative examples are gathered in Section 5. In Section 6, we address the question of injectivity of composition operators. In Section 7, we describe the polar decomposition of a composition operator. Next, in Sections 8 and 9, we characterize normal, quasinormal and formally normal composition operators. Finally, in Section 10, we investigate composition operators which generate Stieltjes moment sequences. We conclude the paper with two appendices. In Appendix A we gather particular properties of  $L^2$ -spaces exploited throughout the paper. Appendix B is mostly devoted to the operator of conditional expectation which plays an essential role in our investigations.

*Caution.* All measure spaces being considered in this paper, except for Appendices A and B, are assumed to be  $\sigma$ -finite.

## 2. Preliminaries

Denote by  $\mathbb{C}$ ,  $\mathbb{R}$  and  $\mathbb{R}_+$  the sets of complex numbers, real numbers and non-negative real numbers, respectively. We write  $\mathbb{Z}_+$  for the set of all nonnegative integers, and  $\mathbb{N}$  for the set of all positive integers. The characteristic function of a subset  $\Delta$  of a set  $X$  will be denoted by  $\chi_\Delta$ . We write  $\Delta \triangle \Delta' = (\Delta \setminus \Delta') \cup (\Delta' \setminus \Delta)$  for subsets  $\Delta$  and  $\Delta'$  of  $X$ . Given a sequence  $\{\Delta_n\}_{n=1}^\infty$  of subsets of  $X$  and a subset  $\Delta$  of  $X$  such that  $\Delta_n \subseteq \Delta_{n+1}$  for every  $n \in \mathbb{N}$ , and  $\Delta = \bigcup_{n=1}^\infty \Delta_n$ , we write  $\Delta_n \nearrow \Delta$  (as  $n \rightarrow \infty$ ). Denote by  $\text{card}(X)$  the cardinal number of  $X$ . If  $X$  is a topological space, then  $\mathfrak{B}(X)$  stands for the  $\sigma$ -algebra of Borel subsets of  $X$ .

Let  $A$  be an operator in a complex Hilbert space  $\mathcal{H}$  (all operators considered in this paper are linear). Denote by  $\mathcal{D}(A)$ ,  $\mathcal{N}(A)$ ,  $\mathcal{R}(A)$ ,  $\bar{A}$  and  $A^*$  the domain, the kernel, the range, the closure and the adjoint of  $A$  (in case they exist). If  $A$  is closed and densely defined, then there exists a unique partial isometry  $U$  on  $\mathcal{H}$  such that  $A = U|A|$  and  $\mathcal{N}(U) = \mathcal{N}(A)$ , where  $|A|$  stands for the square root of  $A^*A$  (cf. [3, Section 8.1]). Set  $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$ . Members of  $\mathcal{D}^\infty(A)$  are called  $C^\infty$ -vectors of  $A$ . Denote by  $\|\cdot\|_A$  the *graph norm* of  $A$ , i.e.,

$$\|f\|_A^2 := \|f\|^2 + \|Af\|^2, \quad f \in \mathcal{D}(A).$$

Given  $n \in \mathbb{Z}_+$ , we define the norm  $\|\cdot\|_{A,n}$  on  $\mathcal{D}(A^n)$  by

$$\|f\|_{A,n}^2 := \sum_{j=0}^n \|A^j f\|^2, \quad f \in \mathcal{D}(A^n).$$

Clearly, for every  $n \in \mathbb{N}$ ,  $(\mathcal{D}(A^n), \|\cdot\|_{A^n})$  and  $(\mathcal{D}(A^n), \|\cdot\|_{A,n})$  are inner product spaces (with standard inner products). A vector subspace  $\mathcal{E}$  of  $\mathcal{D}(A)$  is called a *core* for  $A$  if  $\mathcal{E}$  is dense in  $\mathcal{D}(A)$  with respect to the graph norm of  $A$ . Denote by  $I$  the identity operator on  $\mathcal{H}$ .

By applying Propositions 2.1 and 3.2, one may obtain a criterion for closedness of a linear combination of composition operators.

**PROPOSITION 2.1.** *Let  $A_1, \dots, A_n$  be closed operators in  $\mathcal{H}$  ( $n \in \mathbb{N}$ ). Then  $\sum_{j=1}^n A_j$  is closed if and only if there exists  $c \in \mathbb{R}_+$  such that*

$$\sum_{j=1}^n \|A_j f\|^2 \leq c \left( \|f\|^2 + \left\| \sum_{j=1}^n A_j f \right\|^2 \right), \quad f \in \bigcap_{j=1}^n \mathcal{D}(A_j). \quad (2.1)$$

**PROOF.** Define the vector space  $\mathcal{X} = \bigcap_{j=1}^n \mathcal{D}(A_j)$  and the norm  $\|\cdot\|_*$  on  $\mathcal{X}$  by  $\|f\|_*^2 = \|f\|^2 + \sum_{j=1}^n \|A_j f\|^2$  for  $f \in \mathcal{X}$ . Since the operators  $A_1, \dots, A_n$  are closed, we deduce that  $(\mathcal{X}, \|\cdot\|_*)$  is a Hilbert space. Recall that  $A := \sum_{j=1}^n A_j$  is closed if and only if  $(\mathcal{X}, \|\cdot\|_A)$  is a Hilbert space. Since the identity map from  $(\mathcal{X}, \|\cdot\|_*)$  to  $(\mathcal{X}, \|\cdot\|_A)$  is continuous, we conclude from the inverse mapping theorem that  $(\mathcal{X}, \|\cdot\|_A)$  is a Hilbert space if and only if (2.1) holds for some  $c \in \mathbb{R}_+$ .  $\square$

A densely defined operator  $N$  in  $\mathcal{H}$  is said to be *normal* if  $N$  is closed and  $N^*N = NN^*$  (or equivalently if and only if  $\mathcal{D}(N) = \mathcal{D}(N^*)$  and  $\|Nf\| = \|N^*f\|$  for all  $f \in \mathcal{D}(N)$ , see [55, Proposition, p. 125]). We say that a densely defined operator  $A$  in  $\mathcal{H}$  is *formally normal* if  $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$  and  $\|Af\| = \|A^*f\|$  for all  $f \in \mathcal{D}(A)$  (cf. [12, 2]). A densely defined operator  $A$  in  $\mathcal{H}$  is called *hyponormal* if  $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$  and  $\|A^*f\| \leq \|Af\|$  for all  $f \in \mathcal{D}(A)$  (cf. [27, 35, 53]). Clearly, a closed densely defined operator  $A$  in  $\mathcal{H}$  is normal if and only if both operators  $A$  and  $A^*$  are hyponormal. It is well-known that normality implies formal normality and formal normality implies hyponormality, but none of these implications can be reversed in general. We say that a densely defined operator  $S$  in  $\mathcal{H}$  is *subnormal* if there exist a complex Hilbert space  $\mathcal{K}$  and a normal operator  $N$  in  $\mathcal{K}$  such that  $\mathcal{H} \subseteq \mathcal{K}$  (isometric embedding),  $\mathcal{D}(S) \subseteq \mathcal{D}(N)$  and  $Sf = Nf$  for all  $f \in \mathcal{D}(S)$ .

The members of the next class are related to subnormal operators. A closed densely defined operator  $A$  in  $\mathcal{H}$  is said to be *quasinormal* if  $A$  commutes with the spectral measure  $E_{|A|}$  of  $|A|$ , i.e.,  $E_{|A|}(\Delta)A \subseteq AE_{|A|}(\Delta)$  for all  $\Delta \in \mathfrak{B}(\mathbb{R}_+)$  (cf. [4, 48]). In view of [48, Proposition 1], a closed densely defined operator  $A$  in  $\mathcal{H}$  is quasinormal if and only if  $U|A| \subseteq |A|U$ , where  $A = U|A|$  is the polar decomposition of  $A$ . This combined with [3, Theorem 8.1.5] shows that if  $A$  is a normal operator, then  $A$  is quasinormal and  $\mathcal{N}(A) = \mathcal{N}(A^*)$ . In turn, quasinormality together with the inclusion  $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$  characterizes normality. This result can be found in [52]. For the reader's convenience, we include its proof.

**THEOREM 2.2.** *An operator  $A$  in  $\mathcal{H}$  is normal if and only if  $A$  is quasinormal and  $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$ . Moreover, if  $A$  is normal, then  $\mathcal{N}(A) = \mathcal{N}(A^*)$ .*

**PROOF.** In view of the above discussion it is enough to prove the sufficiency. First we show that if  $A$  is quasinormal and  $A = U|A|$  is its polar decomposition, then  $U|A| = |A|U$ . Indeed, by [48, Proposition 1],  $U|A| \subseteq |A|U$ . Taking adjoints, we get  $U^*|A| \subseteq |A|U^*$ , which implies that  $U^*(\mathcal{D}(|A|)) \subseteq \mathcal{D}(|A|)$ . Hence, if  $f \in \mathcal{D}(|A|U)$ , then  $U^*Uf \in \mathcal{D}(|A|)$ . Since  $I - U^*U$  is the orthogonal projection of  $\mathcal{H}$  onto

$\mathcal{N}(|A|)$ , we conclude that  $f = U^*Uf + (I - U^*U)f \in \mathcal{D}(|A|)$ . This shows that  $\mathcal{D}(|A|U) \subset \mathcal{D}(U|A|)$ , which implies that  $U|A| = |A|U$ .

Now suppose that  $A$  is quasinormal and  $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$ . Since the operators  $P := UU^*$  and  $P^\perp := (I - P)$  are the orthogonal projections of  $\mathcal{H}$  onto  $\overline{\mathcal{R}(A)}$  and  $\mathcal{N}(A^*)$ , respectively, we infer from the inclusion  $\mathcal{N}(A^*) \subset \mathcal{N}(A)$  that

$$\mathcal{R}(P^\perp) \subset \mathcal{N}(A) = \mathcal{N}(|A|) \subset \mathcal{D}(|A|^2). \quad (2.2)$$

It follows from  $U|A| = |A|U$  and  $A^* = |A|U^*$  that

$$AA^* = U|A|^2U^* = |A|^2P. \quad (2.3)$$

We will show that

$$|A|^2P = |A|^2. \quad (2.4)$$

Indeed, if  $f \in \mathcal{H}$ , then, by (2.2) and the equality  $f = Pf + P^\perp f$ , we see that  $Pf \in \mathcal{D}(|A|^2)$  if and only if  $f \in \mathcal{D}(|A|^2)$ . This implies that  $\mathcal{D}(|A|^2P) = \mathcal{D}(|A|^2)$ . Using (2.2) again, we see that  $|A|^2f = |A|^2Pf$  for every  $f \in \mathcal{D}(|A|^2)$ . Hence the equality (2.4) is valid. Combining (2.3) with (2.4), we get  $AA^* = A^*A$ .

The “moreover” part is well-known and easy to prove.  $\square$

Recall that quasinormal operators are subnormal (see [4, Theorem 1] and [48, Theorem 2]). The reverse implication does not hold in general. Clearly, subnormal operators are hyponormal, but not reversely. It is worth pointing out that formally normal operators may not be subnormal (cf. [13, 40, 46]).

A finite complex matrix  $[c_{i,j}]_{i,j=0}^n$  is said to be *nonnegative* if

$$\sum_{i,j=0}^n c_{i,j} \alpha_i \bar{\alpha}_j \geq 0, \quad \alpha_0, \dots, \alpha_n \in \mathbb{C}.$$

If this is the case, then we write  $[c_{i,j}]_{i,j=0}^n \geq 0$ . A sequence  $\{\gamma_n\}_{n=0}^\infty$  of real numbers is said to be a *Stieltjes* moment sequence if there exists a positive Borel measure  $\rho$  on  $\mathbb{R}_+$  such that

$$\gamma_n = \int_{\mathbb{R}_+} s^n d\rho(s), \quad n \in \mathbb{Z}_+.$$

A sequence  $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$  is said to be *positive definite* if for every  $n \in \mathbb{Z}_+$ ,  $[\gamma_{i+j}]_{i,j=0}^n \geq 0$ . By the Stieltjes theorem (see [1, Theorem 6.2.5]), we have

$$\begin{aligned} &\text{a sequence } \{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R} \text{ is a Stieltjes moment sequence if and only if} \\ &\text{the sequences } \{\gamma_n\}_{n=0}^\infty \text{ and } \{\gamma_{n+1}\}_{n=0}^\infty \text{ are positive definite.} \end{aligned} \quad (2.5)$$

**DEFINITION 2.3.** We say that an operator  $S$  in  $\mathcal{H}$  *generates Stieltjes moment sequences* if  $\mathcal{D}^\infty(S)$  is dense in  $\mathcal{H}$  and  $\{\|S^n f\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $f \in \mathcal{D}^\infty(S)$ .

It is well-known that if  $S$  is subnormal, then  $\{\|S^n f\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $f \in \mathcal{D}^\infty(S)$  (see [7, Proposition 3.2.1]; see also Proposition 2.4 below). Hence, if  $\mathcal{D}^\infty(S)$  is dense in  $\mathcal{H}$  and  $S$  is subnormal, then  $S$  generates Stieltjes moment sequences. It turns out that the converse implication does not hold in general (see [7, Section 3.2]).

The following can be proved analogously to [7, Proposition 3.2.1] by using (2.5).

**PROPOSITION 2.4.** *If  $S$  is a subnormal operator in  $\mathcal{H}$ , then the following two assertions hold:*

- (i)  $[\|S^{i+j}f\|^2]_{i,j=0}^n \geq 0$  for all  $f \in \mathcal{D}(S^{2n})$  and  $n \in \mathbb{Z}_+$ ,
- (ii)  $[\|S^{i+j+1}f\|^2]_{i,j=0}^n \geq 0$  for all  $f \in \mathcal{D}(S^{2n+1})$  and  $n \in \mathbb{Z}_+$ .

For the reader's convenience, we state a theorem which is occasionally called the Mittag-Leffler theorem (cf. [41, Lemma 1.1.2]).

**THEOREM 2.5.** *Let  $\{\mathcal{E}_n\}_{n=0}^\infty$  be a sequence of Banach spaces such that for every  $n \in \mathbb{Z}_+$ ,  $\mathcal{E}_{n+1}$  is a vector subspace of  $\mathcal{E}_n$ ,  $\mathcal{E}_{n+1}$  is dense in  $\mathcal{E}_n$  and the embedding map of  $\mathcal{E}_{n+1}$  into  $\mathcal{E}_n$  is continuous. Then  $\bigcap_{n=0}^\infty \mathcal{E}_n$  is dense in each space  $\mathcal{E}_k$ ,  $k \in \mathbb{Z}_+$ .*

### 3. Basic properties of composition operators

From now on, except for Appendices A and B,  $(X, \mathcal{A}, \mu)$  always stands for a  $\sigma$ -finite measure space. We shall abbreviate the expressions “almost everywhere with respect to  $\mu$ ” and “for  $\mu$ -almost every  $x$ ” to “a.e.  $[\mu]$ ” and “for  $\mu$ -a.e.  $x$ ”, respectively. As usual,  $L^2(\mu) = L^2(X, \mathcal{A}, \mu)$  denotes the Hilbert space of all square integrable complex functions on  $X$ . The norm and the inner product of  $L^2(\mu)$  are denoted by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$ , respectively. Let  $\phi$  be an  $\mathcal{A}$ -measurable transformation<sup>1</sup> of  $X$ , i.e.,  $\phi^{-1}(\Delta) \in \mathcal{A}$  for all  $\Delta \in \mathcal{A}$ . Denote by  $\mu \circ \phi^{-1}$  the positive measure on  $\mathcal{A}$  given by  $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$  for all  $\Delta \in \mathcal{A}$ . We say that  $\phi$  is *nonsingular* if  $\mu \circ \phi^{-1}$  is absolutely continuous with respect to  $\mu$ . It is easily seen that if  $\phi$  is nonsingular, then the mapping  $C_\phi: L^2(\mu) \supseteq \mathcal{D}(C_\phi) \rightarrow L^2(\mu)$  given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu): f \circ \phi \in L^2(\mu)\} \text{ and } C_\phi f = f \circ \phi \text{ for } f \in \mathcal{D}(C_\phi), \quad (3.1)$$

is well-defined and linear. Such an operator is called a *composition operator* induced by  $\phi$ ; the transformation  $\phi$  will be referred to as a *symbol* of  $C_\phi$ . Note that if the operator  $C_\phi$  given by (3.1) is well-defined, then the transformation  $\phi$  is nonsingular.

*Convention.* For the remainder of this paper, whenever  $C_\phi$  is mentioned the transformation  $\phi$  is assumed to be nonsingular.

If  $\phi$  is nonsingular, then by the Radon-Nikodym theorem there exists a unique (up to sets of measure zero)  $\mathcal{A}$ -measurable function  $h_\phi: X \rightarrow [0, \infty]$  such that

$$\mu \circ \phi^{-1}(\Delta) = \int_\Delta h_\phi d\mu, \quad \Delta \in \mathcal{A}. \quad (3.2)$$

Here and later on  $\phi^n$  stands for the  $n$ -fold composition of  $\phi$  with itself if  $n \geq 1$  and  $\phi^0$  for the identity transformation of  $X$ . We also write  $\phi^{-n}(\Delta) := (\phi^n)^{-1}(\Delta)$  for  $\Delta \in \mathcal{A}$  and  $n \in \mathbb{Z}_+$ . Note that  $h_{\phi^0} = 1$  a.e.  $[\mu]$ . It is clear that the composition  $\phi_1 \circ \cdots \circ \phi_n$  of finitely many nonsingular transformations  $\phi_1, \dots, \phi_n$  of  $X$  is a nonsingular transformation and

$$C_{\phi_n} \cdots C_{\phi_1} \subseteq C_{\phi_1 \circ \cdots \circ \phi_n}, \quad n \in \mathbb{N}. \quad (3.3)$$

Now we construct an  $\mathcal{A}$ -measurable transformation  $\phi$  of  $X$  such that  $\phi$  is not nonsingular while  $\phi^2$  is nonsingular.

**EXAMPLE 3.1.** Set  $X = \{0\} \cup \{1\} \cup [2, 3]$ . Let  $\mathcal{A} = \{\Delta \cap X: \Delta \in \mathfrak{B}(\mathbb{R}_+)\}$ . Define the finite Borel measure  $\mu$  on  $X$  by

$$\mu(\Delta) = \chi_\Delta(0) + \chi_\Delta(1) + m(\Delta \cap [2, 3]), \quad \Delta \in \mathcal{A},$$

where  $m$  stands for the Lebesgue measure on  $\mathbb{R}$ . Let  $\phi$  be an  $\mathcal{A}$ -measurable transformation of  $X$  given by  $\phi(0) = 2$ ,  $\phi(1) = 1$  and  $\phi(x) = 1$  for  $x \in [2, 3]$ . Since

<sup>1</sup> By a transformation of  $X$  we understand a map from  $X$  to  $X$ .

$\mu(\{2\}) = 0$  and  $(\mu \circ \phi^{-1})(\{2\}) = 1$ , we see that  $\phi$  is not nonsingular. However,  $\phi^2$  is nonsingular because  $\phi^2(x) = 1$  for all  $x \in X$  and  $\mu(\{1\}) > 0$ .

Suppose that  $\phi$  is a nonsingular transformation of  $X$ . In view of the measure transport theorem ([22, Theorem C, p. 163]), we have

$$\int_X |f \circ \phi|^2 d\mu = \int_X |f|^2 h_\phi d\mu \text{ for every } \mathcal{A}\text{-measurable function } f: X \rightarrow \mathbb{C}. \quad (3.4)$$

This implies that

$$\mathcal{D}(C_\phi) = L^2((1 + h_\phi) d\mu), \quad \|f\|_{C_\phi}^2 = \int_X |f|^2 (1 + h_\phi) d\mu, \quad (3.5)$$

$$\mathcal{D}(C_\phi^n) = L^2\left(\left(\sum_{j=0}^n h_{\phi^j}\right) d\mu\right), \quad \|f\|_{C_{\phi,n}}^2 = \int_X |f|^2 \left(\sum_{j=0}^n h_{\phi^j}\right) d\mu, \quad n \in \mathbb{Z}_+. \quad (3.6)$$

Moreover, if  $\phi_1, \dots, \phi_n$  are nonsingular transformations of  $X$  ( $n \in \mathbb{N}$ ), then

$$\mathcal{D}(C_{\phi_n} \cdots C_{\phi_1}) = L^2\left(\left(1 + \sum_{j=1}^n h_{\phi_1 \circ \dots \circ \phi_j}\right) d\mu\right). \quad (3.7)$$

The following proposition is somewhat related to [17, p. 664] and [11, Lemma 6.1].

**PROPOSITION 3.2.** *Let  $\phi$  be a nonsingular transformation of  $X$ . Then  $C_\phi$  is a closed operator and*

$$\overline{\mathcal{D}(C_\phi)} = \chi_{F_\phi} L^2(\mu) \text{ with } F_\phi = \{x \in X : h_\phi(x) < \infty\}. \quad (3.8)$$

Moreover, the following conditions are equivalent:

- (i)  $C_\phi$  is densely defined,
- (ii)  $h_\phi < \infty$  a.e.  $[\mu]$ ,
- (iii) the measure  $\mu \circ \phi^{-1}$  is  $\sigma$ -finite.

**PROOF.** Applying (3.5), we get  $C_\phi = \overline{C_\phi}$  and  $\overline{\mathcal{D}(C_\phi)} \subseteq \chi_{F_\phi} L^2(\mu)$ . To prove the opposite inclusion  $\chi_{F_\phi} L^2(\mu) \subseteq \overline{\mathcal{D}(C_\phi)}$ , take  $f \in L^2(\mu)$  such that  $f|_{X \setminus F_\phi} = 0$  a.e.  $[\mu]$ , and set  $X_n = \{x \in X : h_\phi(x) \leq n\}$  for  $n \in \mathbb{N}$ . Noting that  $X_n \nearrow F_\phi$  as  $n \rightarrow \infty$ , we see that  $\int_X |\chi_{X_n} f|^2 (1 + h_\phi) d\mu < \infty$  for all  $n \in \mathbb{N}$ , and  $\lim_{n \rightarrow \infty} \int_X |f - \chi_{X_n} f|^2 d\mu = 0$ , which completes the proof of (3.8).

(i)  $\Leftrightarrow$  (ii) Employ (3.8).

(ii)  $\Leftrightarrow$  (iii) Apply (3.2) and the assumption that  $\mu$  is  $\sigma$ -finite.  $\square$

**COROLLARY 3.3.** *Suppose that  $\phi_1, \dots, \phi_n$  are nonsingular transformations of  $X$  and  $\lambda_1, \dots, \lambda_n$  are nonzero complex numbers ( $n \in \mathbb{N}$ ). Then  $\sum_{j=1}^n \lambda_j C_{\phi_j}$  is densely defined if and only if  $C_{\phi_k}$  is densely defined for every  $k = 1, \dots, n$ .*

**PROOF.** By (3.5),  $\mathcal{D}(\sum_{j=1}^n \lambda_j C_{\phi_j}) = L^2((1 + \sum_{j=1}^n h_{\phi_j}) d\mu)$ , and thus the “if” part follows from Proposition 3.2 and Lemma A.1. The “only if” part is obvious.  $\square$

#### 4. Products of composition operators

First we give necessary and sufficient conditions for a product of composition operators to be densely defined.

**PROPOSITION 4.1.** *Let  $\phi_1, \dots, \phi_n$  be nonsingular transformations of  $X$  ( $2 \leq n < \infty$ ). Then the following assertions hold:*

- (i)  $C_{\phi_n} \cdots C_{\phi_1}$  is a closable operator,

(ii)  $C_{\phi_n} \cdots C_{\phi_1}$  is densely defined if and only if  $C_{\phi_1 \circ \cdots \circ \phi_k}$  is densely defined for every  $k = 1, \dots, n$ ,

(iii) if  $C_{\phi_{n-1}} \cdots C_{\phi_1}$  is densely defined, then

$$C_{\phi_1 \circ \cdots \circ \phi_k} = \overline{C_{\phi_k} \cdots C_{\phi_1}}, \quad k = 1, \dots, n, \quad (4.1)$$

(iv) if  $C_{\phi_1 \circ \cdots \circ \phi_n}$  is densely defined, then so is the operator  $C_{\phi_n}$ ,

(v) if  $C_{\phi_n} \cdots C_{\phi_1}$  is densely defined, then so are the operators  $C_{\phi_1}, \dots, C_{\phi_n}$ .

PROOF. (i) Apply (3.3) and Proposition 3.2.

(ii) To prove the “if” part, assume that  $C_{\phi_1 \circ \cdots \circ \phi_k}$  is densely defined for  $k = 1, \dots, n$ . It follows from Proposition 3.2 that  $h_{\phi_1 \circ \cdots \circ \phi_k} < \infty$  a.e.  $[\mu]$  for  $k = 1, \dots, n$ . Applying (3.7) and Lemma A.1 to  $\rho_1 \equiv 1$  and  $\rho_2 = 1 + \sum_{j=1}^n h_{\phi_1 \circ \cdots \circ \phi_j}$  we get  $\overline{\mathcal{D}(C_{\phi_n} \cdots C_{\phi_1})} = L^2(\mu)$ . The “only if” part follows from (3.3) and the fact that the operators  $C_{\phi_k} \cdots C_{\phi_1}$ ,  $k = 1, \dots, n$ , are densely defined.

(iii) It follows from (ii) and Proposition 3.2 that  $h := \sum_{j=1}^{n-1} h_{\phi_1 \circ \cdots \circ \phi_j} < \infty$  a.e.  $[\mu]$ . Set  $Y = \{x \in X : h_{\phi_1 \circ \cdots \circ \phi_n}(x) < \infty\}$  and  $\mathcal{A}_Y = \{\Delta \in \mathcal{A} : \Delta \subseteq Y\}$ . Equip  $\mathcal{D}(C_{\phi_1 \circ \cdots \circ \phi_n})$  with the graph norm of  $C_{\phi_1 \circ \cdots \circ \phi_n}$  and note that the mapping

$$\Theta : \mathcal{D}(C_{\phi_1 \circ \cdots \circ \phi_n}) \ni f \mapsto f|_Y \in L^2(Y, \mathcal{A}_Y, (1 + h_{\phi_1 \circ \cdots \circ \phi_n}) d\mu)$$

is a well-defined unitary isomorphism (use (3.5)). It follows from Lemma A.1 that  $L^2(Y, \mathcal{A}_Y, (1 + h + h_{\phi_1 \circ \cdots \circ \phi_n}) d\mu)$  is dense in  $L^2(Y, \mathcal{A}_Y, (1 + h_{\phi_1 \circ \cdots \circ \phi_n}) d\mu)$ . Since, by (3.3) and (3.7),  $\Theta(\mathcal{D}(C_{\phi_n} \cdots C_{\phi_1})) = L^2(Y, \mathcal{A}_Y, (1 + h + h_{\phi_1 \circ \cdots \circ \phi_n}) d\mu)$ , we deduce that  $\overline{C_{\phi_n} \cdots C_{\phi_1}} = C_{\phi_1 \circ \cdots \circ \phi_n}$ . Applying the previous argument to the systems  $(C_{\phi_1}, \dots, C_{\phi_k})$ ,  $k \in \{1, \dots, n-1\}$ , we obtain (4.1).

(iv) It is sufficient to discuss the case of  $n = 2$ . Suppose that  $C_{\phi_1 \circ \phi_2}$  is densely defined. In view of Proposition 3.2, the measure  $\mu \circ (\phi_1 \circ \phi_2)^{-1}$  is  $\sigma$ -finite. Since  $\mu \circ (\phi_1 \circ \phi_2)^{-1} = (\mu \circ \phi_2^{-1}) \circ \phi_1^{-1}$ , we see that the measure  $\mu \circ \phi_2^{-1}$  is  $\sigma$ -finite as well. Applying Proposition 3.2 again, we conclude that  $C_{\phi_2}$  is densely defined.

(v) Apply (ii) and (iv).  $\square$

COROLLARY 4.2. If  $C_{\phi}^{n-1}$  is densely defined for some  $n \in \mathbb{N}$ , then  $\overline{C_{\phi}^n} = C_{\phi^n}$ .

The following is an immediate consequence of (3.7) and Corollary A.4.

PROPOSITION 4.3. If  $\phi_1, \dots, \phi_m$  and  $\psi_1, \dots, \psi_n$  are nonsingular transformations of  $X$ , then  $\mathcal{D}(C_{\phi_n} \cdots C_{\phi_1}) \subseteq \mathcal{D}(C_{\psi_m} \cdots C_{\psi_1})$  if and only if there exists  $c \in \mathbb{R}_+$  such that  $\sum_{j=1}^m h_{\psi_1 \circ \cdots \circ \psi_j} \leq c(1 + \sum_{j=1}^n h_{\phi_1 \circ \cdots \circ \phi_j})$  a.e.  $[\mu]$ .

Now we give necessary and sufficient conditions for a product of composition operators to be closed.

PROPOSITION 4.4. Let  $\phi_1, \dots, \phi_n$  be nonsingular transformations of  $X$  ( $2 \leq n < \infty$ ). Then the following three conditions are equivalent:

- (i)  $C_{\phi_n} \cdots C_{\phi_1} = C_{\phi_1 \circ \cdots \circ \phi_n}$ ,
- (ii)  $\mathcal{D}(C_{\phi_1 \circ \cdots \circ \phi_n}) \subseteq \mathcal{D}(C_{\phi_n} \cdots C_{\phi_1})$ ,
- (iii) there exists  $c \in \mathbb{R}_+$  such that  $\sum_{j=1}^{n-1} h_{\phi_1 \circ \cdots \circ \phi_j} \leq c(1 + h_{\phi_1 \circ \cdots \circ \phi_n})$  a.e.  $[\mu]$ .

Moreover, any of the conditions (i) to (iii) implies that

- (iv)  $C_{\phi_n} \cdots C_{\phi_1}$  is closed.

If  $C_{\phi_{n-1}} \cdots C_{\phi_1}$  is densely defined, then all the conditions (i) to (iv) are equivalent.



PROOF. The equivalence of (i) and (ii) is a direct consequence of (3.3). The equivalence of (ii) and (iii) follows from Proposition 4.3. That (i) implies (iv) follows from Proposition 3.2. Finally, if the product  $C_{\phi_{n-1}} \cdots C_{\phi_1}$  is densely defined, then (iv) implies (i) due to Proposition 4.1 (iii).  $\square$

COROLLARY 4.5. *If  $\phi$  is a nonsingular transformation of  $X$ , then the following assertions hold for all  $n \in \mathbb{N}$ :*

- (i)  $C_{\phi^n}$  is densely defined if and only  $\mathbf{h}_{\phi^n} < \infty$  a.e.  $[\mu]$ ,
- (ii)  $C_{\phi^n}$  is densely defined if and only  $\sum_{j=1}^n \mathbf{h}_{\phi^j} < \infty$  a.e.  $[\mu]$ ,
- (iii)  $C_{\phi^n} = C_{\phi^n}$  if and only if there exists  $c \in \mathbb{R}_+$  such that  $\mathbf{h}_{\phi^k} \leq c(1 + \mathbf{h}_{\phi^n})$  a.e.  $[\mu]$  for  $k = 1, \dots, n$ .

PROOF. Use Propositions 3.2, 4.1 (ii) and 4.4 (for (ii) see also [24, p. 515]).  $\square$

COROLLARY 4.6. *If  $\phi$  is a nonsingular transformation of  $X$  and  $\overline{\mathcal{D}(C_{\phi}^m)} = L^2(\mu)$  for some  $m \in \mathbb{N}$ , then there exists a sequence  $\{X_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$  such that*

- (i)  $X_n \nearrow X$  as  $n \rightarrow \infty$ ,
- (ii)  $\mu(X_n) < \infty$  for all  $n \in \mathbb{N}$ ,
- (iii)  $\sum_{j=1}^m \mathbf{h}_{\phi^j}(x) \leq n$  for  $\mu$ -a.e.  $x \in X_n$  and  $n \in \mathbb{N}$ .

The question of when  $C^\infty$ -vectors of an operator  $A$  in a Hilbert space  $\mathcal{H}$  form a dense subspace of  $\mathcal{H}$  is of independent interest (cf. [39, 28]). If every power of  $A$  is densely defined, then one could expect that  $\mathcal{D}^\infty(A)$  is dense in  $\mathcal{H}$ . This is the case for any closed densely defined operator (even in a Banach space), the resolvent set of which is nonempty<sup>2</sup>. As shown below, this is also the case for composition operators. However, this seems to be not true in general. Dropping the assumption of closedness, we can provide a simple counterexample. Indeed, take an infinite dimensional separable Hilbert space  $\mathcal{H}$ . Then there exists a dense subset  $\{e_n : n \in \mathbb{Z}_+\}$  of  $\mathcal{H}$  which consists of linearly independent vectors. Let  $A$  be the operator in  $\mathcal{H}$  whose domain is the linear span of  $\{e_n : n \in \mathbb{N}\}$  and  $Ae_j = e_{j-1}$  for every  $j \in \mathbb{N}$ . Since  $\{e_n : n \geq k\}$  is dense in  $\mathcal{H}$  for every  $k \in \mathbb{Z}_+$ , we deduce that the operator  $A^n$  is densely defined for every  $n \in \mathbb{Z}_+$ . However,  $\mathcal{D}^\infty(A) = \{0\}$ .

THEOREM 4.7. *If  $\phi$  is a nonsingular transformation of  $X$ , then the following conditions are equivalent:*

- (i)  $\mathcal{D}(C_{\phi}^n)$  is dense in  $L^2(\mu)$  for every  $n \in \mathbb{N}$ ,
- (ii)  $\mathcal{D}^\infty(C_{\phi})$  is dense in  $L^2(\mu)$ ,
- (iii)  $\mathcal{D}^\infty(C_{\phi})$  is a core for  $C_{\phi}^n$  for every  $n \in \mathbb{Z}_+$ ,
- (iv)  $\mathcal{D}^\infty(C_{\phi})$  is dense in  $(\mathcal{D}(C_{\phi}^n), \|\cdot\|_{C_{\phi},n})$  for every  $n \in \mathbb{Z}_+$ .

PROOF. The implications (iv) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious.

(i) $\Rightarrow$ (iv) In view of Corollary 4.5 (ii),  $0 \leq \mathbf{h}_{\phi^n} < \infty$  a.e.  $[\mu]$  for all  $n \in \mathbb{N}$ . Given  $n \in \mathbb{Z}_+$  we denote by  $\mathcal{H}_n$  the inner product space  $(\mathcal{D}(C_{\phi}^n), \|\cdot\|_{C_{\phi},n})$ . It follows from (3.6) that  $\mathcal{H}_n$  is a Hilbert space which coincides with  $L^2((\sum_{j=0}^n \mathbf{h}_{\phi^j}) d\mu)$ . Hence, in view of Lemma A.1,  $\mathcal{H}_{n+1}$  is a dense subspace of  $\mathcal{H}_n$ . Clearly, the embedding map of  $\mathcal{H}_{n+1}$  into  $\mathcal{H}_n$  is continuous. Applying Theorem 2.5 to the sequence  $\{\mathcal{H}_n\}_{n=0}^{\infty}$ , we conclude that  $\mathcal{D}^\infty(C_{\phi}) = \bigcap_{i=0}^{\infty} \mathcal{H}_i$  is dense in  $\mathcal{D}(C_{\phi}^n)$  with respect to the norm  $\|\cdot\|_{C_{\phi},n}$  for every  $n \in \mathbb{Z}_+$ . This completes the proof.  $\square$

<sup>2</sup> This can be deduced from the fact that the intersection of ranges of all powers of a bounded operator which has dense range is dense in the underlying space.



Regarding Theorem 4.7, we mention the following surprising fact which can be deduced from [39, Theorem 4.5] by using Theorem 2.5 and [39, Corollaries 1.2 and 1.4].

**THEOREM 4.8.** *Let  $A$  be an unbounded selfadjoint operator in a complex Hilbert space  $\mathcal{H}$  and let  $\mathfrak{N}$  be a (possibly empty) subset of  $\mathbb{N} \setminus \{1\}$  such that  $\mathbb{N} \setminus \mathfrak{N}$  is infinite. Then there exists a closed symmetric operator  $T$  in  $\mathcal{H}$  such that  $T \subseteq A$ ,  $\mathcal{D}^\infty(T)$  is dense in  $\mathcal{H}$  and for every  $k \in \mathbb{N}$ ,  $\mathcal{D}^\infty(T)$  is a core for  $T^k$  if and only if  $k \in \mathbb{N} \setminus \mathfrak{N}$ .*

## 5. Examples

We begin by showing that Corollary 4.2 is no longer true if the assumption that  $C_\phi^{n-1}$  is densely defined is dropped.

**EXAMPLE 5.1.** We will demonstrate that there is a nonsingular transformation  $\phi$  such that  $C_\phi$  is densely defined,  $C_{\phi^j}$  and  $C_\phi^j$  are not densely defined for every  $j \in \{2, 3, \dots\}$ , and  $\overline{C_\phi^3} \subsetneq C_{\phi^3}$  (however, by Corollary 4.2,  $\overline{C_\phi^2} = C_{\phi^2}$ ). For this, we will re-examine Example 4.2 given in [24]. Suppose that  $\{a_i\}_{i=0}^\infty$ ,  $\{b_i\}_{i=0}^\infty$  and  $\{c_{i,j}\}_{i,j=0}^\infty$  are disjoint sets of distinct elements. Set  $X = \{a_i\}_{i=0}^\infty \cup \{b_i\}_{i=0}^\infty \cup \{c_{i,j}\}_{i,j=0}^\infty$  and  $\mathcal{A} = 2^X$ . Let  $\mu$  be a unique  $\sigma$ -finite measure on  $\mathcal{A}$  determined by

$$\mu(\{x\}) = \begin{cases} 1 & \text{if } x = a_i \text{ for some } i \in \mathbb{Z}_+, \\ \frac{1}{2^{i+1}} & \text{if } x = b_i \text{ for some } i \in \mathbb{Z}_+, \\ \frac{1}{2^{j+1}} & \text{if } x = c_{i,j} \text{ for some } i, j \in \mathbb{Z}_+. \end{cases}$$

Define a nonsingular transformation  $\phi$  of  $X$  by

$$\phi(x) = \begin{cases} a_{i+1} & \text{if } x = a_i \text{ for some } i \in \mathbb{Z}_+, \\ a_0 & \text{if } x = b_i \text{ for some } i \in \mathbb{Z}_+, \\ b_i & \text{if } x = c_{i,j} \text{ for some } i, j \in \mathbb{Z}_+. \end{cases}$$

Then  $h_\phi < \infty$  a.e.  $[\mu]$ , and thus by Proposition 3.2 the operator  $C_\phi$  is densely defined. Since  $h_{\phi^2}(a_0) = \infty$ , we infer from Proposition 3.2 that  $C_{\phi^2}$  is not densely defined. It follows from (3.7) that  $\mathcal{D}(C_\phi^3) = L^2((1 + h_\phi + h_{\phi^2} + h_{\phi^3})d\mu)$ . This and  $h_{\phi^2}(a_0) = \infty$  imply that  $f(a_0) = 0$  for every  $f \in \mathcal{D}(C_\phi^3)$ . Since the convergence in the graph norm is stronger than the pointwise convergence, we deduce that  $f(a_0) = 0$  for every  $f \in \mathcal{D}(\overline{C_\phi^3})$ . As  $\mathcal{D}(C_{\phi^3}) = L^2((1 + h_{\phi^3})d\mu)$  (cf. (3.5)) and  $h_{\phi^3}(a_0) = 0$  (because  $\phi^{-3}(\{a_0\}) = \emptyset$ ), we see that  $\chi_{\{a_0\}} \in \mathcal{D}(C_{\phi^3}) \setminus \mathcal{D}(\overline{C_\phi^3})$ . Finally, arguing as above and using the fact that  $h_{\phi^{j+2}}(a_j) = \infty$  for every  $j \in \mathbb{Z}_+$ , we conclude that  $C_{\phi^j}$  is not densely defined for every  $j \in \{2, 3, \dots\}$ . As a consequence,  $C_\phi^j$  is not densely defined for every  $j \in \{2, 3, \dots\}$ .

The composition operator  $C_\phi$  constructed in Example 5.1 is densely defined, its square is not densely defined, however  $\dim \mathcal{D}(C_\phi^n) = \infty$  for all  $n \in \mathbb{N}$  (because  $\chi_{\{a_i\}} \in \mathcal{D}(C_\phi^n)$  for all  $i \geq n-1$ ). In fact, there are more pathological examples.

**EXAMPLE 5.2.** It was proved in [26, Theorem 4.2] that there exists a hyponormal weighted shift  $S$  on a rootless and leafless directed tree with positive weights whose square has trivial domain. By [25, Lemma 4.3.1],  $S$  is unitarily equivalent to a composition operator  $C$ . As a consequence,  $C$  is injective and hyponormal, and  $\mathcal{D}(C^2) = \mathcal{D}^\infty(C) = \{0\}$  (see also [6] for a recent construction).

Regarding Proposition 4.1, we note that it may happen that the operators  $C_{\phi_1}$  and  $C_{\phi_2}$  are densely defined, while the operators  $C_{\phi_1 \circ \phi_2}$  and  $C_{\phi_2} C_{\phi_1}$  are not (even if  $\phi_1 = \phi_2$ , see Example 5.1). Below we will show that for some  $\phi_1$  and  $\phi_2$  the composition operator  $C_{\phi_1 \circ \phi_2}$  is densely defined (even bounded), while  $C_{\phi_1}$  is not.

EXAMPLE 5.3. Set  $X = \mathbb{Z}_+$  and  $\mathcal{A} = 2^X$ . Let  $\mu$  be the counting measure on  $X$  and let  $\phi_1$  and  $\phi_2$  be the nonsingular transformations of  $X$  given by  $\phi_1(2n) = n$ ,  $\phi_1(2n+1) = 0$  and  $\phi_2(n) = 2n$  for  $n \in \mathbb{Z}_+$ . Then  $\phi_1 \circ \phi_2$  is the identity transformation of  $X$ , and hence  $C_{\phi_1 \circ \phi_2}$  is the identity operator on  $L^2(\mu)$ . However, since  $\mu(\phi_1^{-1}(\{0\})) = \infty$ , the measure  $\mu \circ \phi_1^{-1}$  is not  $\sigma$ -finite, and thus by Proposition 3.2 the operator  $C_{\phi_1}$  is not densely defined.

Our next aim is to provide examples showing that the equality  $C_\phi^n = C_{\phi^n}$  which appears in Corollary 4.5 (iii) does not hold in general even if  $\mathcal{D}^\infty(C_\phi)$  is dense in  $L^2(\mu)$  (which is not the case for the operator given in Example 5.1).

EXAMPLE 5.4. Set  $X = \mathbb{N}$  and  $\mathcal{A} = 2^X$ . Let  $\mu$  be a counting measure on  $X$  and let  $\{J_n\}_{n=1}^\infty$  be a partition of  $X$ . Define a nonsingular transformation  $\phi$  of  $X$  by  $\phi(x) = \min J_{n^2}$  for  $x \in J_n$  and  $n \in \mathbb{N}$ . Set  $\mathbb{N}_s = \{n^2 : n \in \mathbb{N}\}$  and note that

$$X = \{1\} \sqcup \bigsqcup_{q \in \mathbb{N} \setminus \mathbb{N}_s} \{q^{2^n} : n \in \mathbb{Z}_+\}, \quad (5.1)$$

where all terms in (5.1) are pairwise disjoint (they are equivalence classes under the equivalence relation  $\sim$  given by:  $p \sim q$  if and only if  $p^{2^m} = q^{2^n}$  for some  $m, n \in \mathbb{Z}_+$ ). Since  $h_{\phi^j}(x) = \text{card}(\phi^{-j}(\{x\}))$  for  $x \in X$  and  $j \in \mathbb{N}$ , we infer from (5.1) that for all  $j \in \mathbb{N}$  and  $x \in X$  ( $m$  appearing below varies over the set of integers)

$$h_{\phi^j}(x) = \begin{cases} \text{card}(J_1) & \text{if } x = \min J_1, \\ \text{card}(J_{q^{2^{m-j}}}) & \text{if } x = \min J_{q^{2^m}} \text{ with } q \in \mathbb{N} \setminus \mathbb{N}_s \text{ and } m \geq j, \\ 0 & \text{otherwise.} \end{cases} \quad (5.2)$$

By (5.1), (5.2), Proposition 3.2 and Theorem 4.7, the following are equivalent:

- $\text{card}(J_k) < \aleph_0$  for every  $k \in \mathbb{N}$ ,
- $C_\phi$  is densely defined,
- $C_\phi^n$  is densely defined for some  $n \in \mathbb{N}$ ,
- $C_\phi^n$  is densely defined for every  $n \in \mathbb{N}$ ,
- $\mathcal{D}^\infty(C_\phi)$  is dense in  $L^2(\mu)$ .

It follows from (5.2) and Proposition 4.4 that for a given integer  $n \geq 2$ ,  $C_\phi^n$  is closed if and only if there exists  $c \in \mathbb{R}_+$  such that

$$\begin{aligned} \text{card}(J_{q^{2^s}}) &\leq c, \quad s = 0, \dots, n-2, q \in \mathbb{N} \setminus \mathbb{N}_s, \\ \text{card}(J_{q^{2^{s+1}}}) &\leq c(1 + \text{card}(J_{q^{2^s}})), \quad s \in \mathbb{Z}_+, q \in \mathbb{N} \setminus \mathbb{N}_s. \end{aligned}$$

Using this and an induction argument, one can prove that either  $C_\phi^n$  is closed for every integer  $n \geq 1$ , or  $C_\phi^n$  is not closed for every integer  $n \geq 2$ . Summarizing, if we choose a partition  $\{J_i\}_{i=1}^\infty$  of  $X$  such that  $J_n$  is finite for every  $n \in \mathbb{N}$ , and  $\sup\{\text{card}(J_q) : q \in \mathbb{N} \setminus \mathbb{N}_s\} = \aleph_0$  (which is possible), then  $\mathcal{D}^\infty(C_\phi)$  is dense in  $L^2(\mu)$  and  $C_\phi^n$  is not closed for every integer  $n \geq 2$ . On the other hand, if  $\kappa \geq 2$  is any fixed integer and a partition  $\{J_i\}_{i=1}^\infty$  of  $X$  is selected so that  $J_1$  is finite and  $\text{card}(J_{q^{2^n}}) = \kappa^n$  for all  $n \in \mathbb{Z}_+$  and  $q \in \mathbb{N} \setminus \mathbb{N}_s$  (which is also possible), then  $\mathcal{D}^\infty(C_\phi)$  is dense in  $L^2(\mu)$  and  $C_\phi^n$  is closed and unbounded for every  $n \in \mathbb{N}$ .

## 6. Injectivity of $C_\phi$

In this section we provide necessary and sufficient conditions for a composition operator to be injective. The following set plays an important role in our considerations.

$$\mathbf{N}_\phi = \{x \in X : h_\phi(x) = 0\}.$$

The following description of the kernel of  $C_\phi$  follows immediately from (3.4).

PROPOSITION 6.1. *If  $\phi: X \rightarrow X$  is nonsingular, then  $\mathcal{N}(C_\phi) = \chi_{\mathbf{N}_\phi} L^2(\mu)$ .*

PROPOSITION 6.2. *Let  $\phi$  be a nonsingular transformation of  $X$ . Consider the following four conditions:*

- (i)  $\mathcal{N}(C_\phi) = \{0\}$ ,
- (ii)  $\mu(\mathbf{N}_\phi) = 0$ ,
- (iii)  $\chi_{\mathbf{N}_\phi} \circ \phi = \chi_{\mathbf{N}_\phi}$  a.e.  $[\mu]$ ,
- (iv)  $\mathcal{N}(C_\phi) \subseteq \mathcal{N}(C_\phi^*)$ .

*Then the conditions (i), (ii) and (iii) are equivalent. Moreover, if  $C_\phi$  is densely defined, then the conditions (i) to (iv) are equivalent.*

PROOF. (i) $\Leftrightarrow$ (ii) Apply Proposition 6.1 and the  $\sigma$ -finiteness of  $\mu$ .

(ii) $\Rightarrow$ (iii) Since  $\phi$  is nonsingular, we have  $\mu(\mathbf{N}_\phi) = 0$  and  $\mu(\phi^{-1}(\mathbf{N}_\phi)) = 0$ , which implies that  $\mu(\mathbf{N}_\phi \Delta \phi^{-1}(\mathbf{N}_\phi)) = 0$ . The latter is equivalent to (iii).

(iii) $\Rightarrow$ (ii) By the measure transport theorem, we have

$$\mu(\mathbf{N}_\phi) = \int_X \chi_{\mathbf{N}_\phi} d\mu = \int_X \chi_{\mathbf{N}_\phi} \circ \phi d\mu = \int_X \chi_{\mathbf{N}_\phi} h_\phi d\mu = 0.$$

Now suppose that  $C_\phi$  is densely defined.

(i) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (ii) Let  $\{X_n\}_{n=1}^\infty$  be as in Corollary 4.6 (with  $m = 1$ ). Then, by (3.4), we see that  $\chi_{X_n}, \chi_{\mathbf{N}_\phi \cap X_n} \in \mathcal{D}(C_\phi)$  and  $\|C_\phi(\chi_{\mathbf{N}_\phi \cap X_n})\|^2 = \int_{\mathbf{N}_\phi \cap X_n} h_\phi d\mu = 0$  for all  $n \in \mathbb{N}$ , which together with our assumption that  $\mathcal{N}(C_\phi) \subseteq \mathcal{N}(C_\phi^*)$  yields

$$0 = \langle \chi_{\mathbf{N}_\phi \cap X_n}, C_\phi \chi_{X_n} \rangle = \int_{\mathbf{N}_\phi \cap X_n} \chi_{X_n} \circ \phi d\mu = \mu(\mathbf{N}_\phi \cap X_n \cap \phi^{-1}(X_n))$$

for all  $n \in \mathbb{N}$ . Since  $\mathbf{N}_\phi \cap X_n \cap \phi^{-1}(X_n) \nearrow \mathbf{N}_\phi$  as  $n \rightarrow \infty$ , the continuity of measure implies that  $\mu(\mathbf{N}_\phi) = 0$ . This completes the proof.  $\square$

COROLLARY 6.3. *If  $C_\phi$  is hyponormal, then  $\mathcal{N}(C_\phi) = \{0\}$ .*

PROOF. It follows from the definition of hyponormality that  $\mathcal{N}(C_\phi) \subseteq \mathcal{N}(C_\phi^*)$ . This and Proposition 6.2 complete the proof.  $\square$

COROLLARY 6.4. *If  $C_\phi$  is formally normal, then*

$$\mathcal{D}(C_\phi) \cap \mathcal{N}(C_\phi^*) = \{0\}.$$

PROOF. Indeed, if  $f \in \mathcal{D}(C_\phi) \cap \mathcal{N}(C_\phi^*)$ , then  $\|C_\phi f\| = \|C_\phi^* f\| = 0$ , which means that  $f \in \mathcal{N}(C_\phi)$ . Applying Corollary 6.3, completes the proof.  $\square$

It turns out that composition of  $h_\phi$  with  $\phi$  is positive a.e.  $[\mu]$  (see also the proof of [23, Corollary 5]).

PROPOSITION 6.5. *If  $\phi: X \rightarrow X$  is nonsingular, then  $h_\phi \circ \phi > 0$  a.e.  $[\mu]$ .*

PROOF. Note that  $\mu(\phi^{-1}(\mathbf{N}_\phi)) = \int_X \chi_{\mathbf{N}_\phi} \circ \phi \, d\mu = \int_X \chi_{\mathbf{N}_\phi} \mathbf{h}_\phi \, d\mu = 0$ . This combined with  $\phi^{-1}(\mathbf{N}_\phi) = \{x \in X : \mathbf{h}_\phi(\phi(x)) = 0\}$  completes the proof.  $\square$

COROLLARY 6.6. *If  $\phi$  is a nonsingular transformation of  $X$  and  $\mathbf{h}_\phi \circ \phi = \mathbf{h}_\phi$  a.e.  $[\mu]$ , then  $\mathcal{N}(C_\phi) = \{0\}$ .*

PROOF. Apply Propositions 6.1 and 6.5.  $\square$

## 7. The polar decomposition

Given an  $\mathcal{A}$ -measurable function  $u : X \rightarrow \mathbb{C}$ , we denote by  $M_u$  the operator of multiplication by  $u$  in  $L^2(\mu)$  defined by

$$\begin{aligned} \mathcal{D}(M_u) &= \{f \in L^2(\mu) : u \cdot f \in L^2(\mu)\}, \\ M_u f &= u \cdot f, \quad f \in \mathcal{D}(M_u). \end{aligned}$$

The operator  $M_u$  is a normal operator (cf. [3, Section 7.2]).

The polar decomposition of  $C_\phi$  can be explicitly described as follows.

PROPOSITION 7.1. *Suppose that the composition operator  $C_\phi$  is densely defined and  $C_\phi = U|C_\phi|$  is its polar decomposition. Then*

(i)  $|C_\phi| = M_{\mathbf{h}_\phi^{1/2}}$ ,

(ii) *the initial space of  $U$  is given by*<sup>3</sup>

$$\overline{\mathcal{R}(|C_\phi|)} = \{\mathbf{h}_\phi^{1/2} f : f \in L^2(\mathbf{h}_\phi \, d\mu)\}, \quad (7.1)$$

(iii) *the final space of  $U$  is given by*

$$\overline{\mathcal{R}(C_\phi)} = \{f \circ \phi : f \in L^2(\mathbf{h}_\phi \, d\mu)\}, \quad (7.2)$$

(iv) *the partial isometry  $U$  is given by*<sup>4</sup>

$$Ug = \frac{g \circ \phi}{(\mathbf{h}_\phi \circ \phi)^{1/2}}, \quad g \in L^2(\mu), \quad (7.3)$$

(v) *the adjoint  $U^*$  of  $U$  is given by*

$$U^*g = \mathbf{h}_\phi^{1/2} \cdot V^{-1}Pg, \quad g \in L^2(\mu),$$

where  $V : L^2(\mathbf{h}_\phi \, d\mu) \rightarrow \overline{\mathcal{R}(C_\phi)}$  is a unitary operator defined by  $Vf = f \circ \phi$  for  $f \in L^2(\mathbf{h}_\phi \, d\mu)$  and  $P$  is the orthogonal projection of  $L^2(\mu)$  onto  $\overline{\mathcal{R}(C_\phi)}$ .

PROOF. (i) We will show that  $C_\phi^* C_\phi \subseteq M_{\mathbf{h}_\phi}$ . Let  $\{X_n\}_{n=1}^\infty$  be as in Corollary 4.6 (with  $m = 1$ ). Take  $f \in \mathcal{D}(C_\phi^* C_\phi)$  and fix  $n \in \mathbb{N}$ . By (3.5),  $\chi_\Delta \in \mathcal{D}(C_\phi)$  whenever  $\Delta \in \mathcal{A}$  and  $\Delta \subseteq X_n$ . Thus, for every such  $\Delta$ , we have

$$\int_\Delta C_\phi^* C_\phi f \, d\mu = \langle C_\phi^* C_\phi f, \chi_\Delta \rangle = \langle C_\phi f, C_\phi \chi_\Delta \rangle \stackrel{(3.2)}{=} \int_\Delta f \mathbf{h}_\phi \, d\mu.$$

Since both functions  $(C_\phi^* C_\phi f) \chi_{X_n}$  and  $(f \mathbf{h}_\phi) \chi_{X_n}$  are in  $L^1(\mu)$ , we deduce that  $C_\phi^* C_\phi f = f \mathbf{h}_\phi$  a.e.  $[\mu]$  on  $X_n$ . This and  $X_n \nearrow X$  give  $C_\phi^* C_\phi f = f \mathbf{h}_\phi$  a.e.  $[\mu]$ . As a consequence, we have  $C_\phi^* C_\phi \subseteq M_{\mathbf{h}_\phi}$ . Since both are selfadjoint operators, they are equal. Thus  $|C_\phi| = M_{\mathbf{h}_\phi}^{1/2} = M_{\mathbf{h}_\phi^{1/2}}$ .

<sup>3</sup> Note that the mapping  $L^2(\mathbf{h}_\phi \, d\mu) \ni f \mapsto \mathbf{h}_\phi^{1/2} f \in L^2(\mu)$  is an isometry.

<sup>4</sup> Recall that  $\mathbf{h}_\phi \circ \phi > 0$  a.e.  $[\mu]$  (cf. Proposition 6.5).

(ii) By [3, Section 8.1] and Proposition 6.1, we have

$$\overline{\mathcal{R}(|C_\phi|)} = \mathcal{N}(|C_\phi|)^\perp = \mathcal{N}(C_\phi)^\perp = \chi_{X \setminus \mathbb{N}_\phi} L^2(\mu), \quad (7.4)$$

which as easily seen gives (7.1).

(iii) By (3.4) and (ii), the mapping  $W: \overline{\mathcal{R}(|C_\phi|)} \rightarrow L^2(\mu)$  given by

$$W(\mathbf{h}_\phi^{1/2} f) = f \circ \phi, \quad f \in L^2(\mathbf{h}_\phi d\mu). \quad (7.5)$$

is a well-defined isometry. Using (i) we verify that  $W|_{\mathcal{R}(|C_\phi|)} = U|_{\mathcal{R}(|C_\phi|)}$ , which implies that  $\overline{\mathcal{R}(C_\phi)} = \mathcal{R}(U) = \mathcal{R}(W)$ . Hence (iii) holds and, by (7.5), we have

$$U^*(f \circ \phi) = \mathbf{h}_\phi^{1/2} f, \quad f \in L^2(\mathbf{h}_\phi d\mu). \quad (7.6)$$

(iv) Applying the measure transport theorem to the restriction of  $\phi$  to the full  $\mu$ -measure set on which  $\mathbf{h}_\phi \circ \phi$  is positive (cf. Proposition 6.5), we get

$$\int_X \frac{|g \circ \phi|^2}{\mathbf{h}_\phi \circ \phi} d\mu = \int_{X \setminus \mathbb{N}_\phi} |g|^2 d\mu, \quad g \in L^2(\mu). \quad (7.7)$$

This and Proposition 6.1 imply that the mapping  $\tilde{U}: L^2(\mu) \ni g \mapsto \frac{g \circ \phi}{(\mathbf{h}_\phi \circ \phi)^{1/2}} \in L^2(\mu)$  is a contraction such that  $\mathcal{N}(\tilde{U}) = \chi_{\mathbb{N}_\phi} L^2(\mu) = \mathcal{N}(|C_\phi|)$ . Hence, by (7.4) and (7.7),  $\tilde{U}$  is an isometry on  $\overline{\mathcal{R}(|C_\phi|)}$ . Clearly, by (i),  $\tilde{U}|_{C_\phi} g = C_\phi g$  for  $g \in \mathcal{D}(C_\phi) = \mathcal{D}(|C_\phi|)$ , which implies that  $U = \tilde{U}$ .

(v) By (3.4) and (7.2),  $V$  is a well-defined unitary operator. If  $g \in L^2(\mu)$ , then by (iii),  $Pg = f \circ \phi$  a.e.  $[\mu]$  for some  $f \in L^2(\mathbf{h}_\phi d\mu)$ . Thus, by  $\mathcal{N}(U^*) = \mathcal{R}(I - P)$  and (7.6), we have

$$U^*g = U^*Pg = U^*(f \circ \phi) = \mathbf{h}_\phi^{1/2} f = \mathbf{h}_\phi^{1/2} \cdot V^{-1}Pg.$$

This completes the proof.  $\square$

Regarding Proposition 7.1, we note that the formulas for  $|C_\phi|$  and  $\overline{\mathcal{R}(C_\phi)}$  are well-known in the case of bounded composition operators (cf. [23, Lemma 1]). The formula (7.3) has appeared in [10, p. 387] in the context of bounded operators without proof.

**COROLLARY 7.2.** *Suppose that  $C_\phi$  is densely defined and  $g \in L^2(\mu)$ . Then  $g$  belongs to  $\overline{\mathcal{R}(C_\phi)}$  if and only if one of the following equivalent conditions holds<sup>5</sup>:*

- (i) *there is an  $\mathcal{A}$ -measurable function  $f: X \rightarrow \mathbb{C}$  such that  $g = f \circ \phi$  a.e.  $[\mu]$ ,*
- (ii) *there is a  $\phi^{-1}(\mathcal{A})$ -measurable function  $f: X \rightarrow \mathbb{C}$  such that  $g = f$  a.e.  $[\mu]$ ,*
- (iii)  *$g$  is  $(\phi^{-1}(\mathcal{A}))^\mu$ -measurable,*
- (iv) *for every Borel set  $\Delta$  in  $\mathbb{C}$  there exists  $\Delta' \in \mathcal{A}$  such that*

$$\mu(g^{-1}(\Delta) \triangle \phi^{-1}(\Delta')) = 0.$$

*In particular,  $\overline{\mathcal{R}(C_\phi)} = L^2(\mu|_{(\phi^{-1}(\mathcal{A}))^\mu})$ .*

**PROOF.** Apply (3.4), (7.2), (B.1), (B.2) and Lemma B.3.  $\square$

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<sup>5</sup> See Appendix for definitions and notation.

**COROLLARY 7.3.** *If  $C_\phi$  is densely defined, then the map  $V : L^2(\mathbf{h}_\phi d\mu) \rightarrow \overline{\mathcal{R}(C_\phi)}$  given by  $Vf = f \circ \phi$  for  $f \in L^2(\mathbf{h}_\phi d\mu)$  is a well-defined unitary operator such that*

$$\begin{aligned} \mathcal{D}(C_\phi^*) &= \{g \in L^2(\mu) : \mathbf{h}_\phi \cdot V^{-1}Pg \in L^2(\mu)\}, \\ C_\phi^*g &= \mathbf{h}_\phi \cdot V^{-1}Pg, \quad g \in \mathcal{D}(C_\phi^*), \end{aligned} \quad (7.8)$$

where  $P$  is the orthogonal projection of  $L^2(\mu)$  onto  $\overline{\mathcal{R}(C_\phi)} = L^2(\mu|_{(\phi^{-1}(\mathcal{A}))^\mu})$ .

**PROOF.** If  $C_\phi = U|C_\phi|$  is the polar decomposition of  $C_\phi$ , then  $C_\phi^* = |C_\phi|U^*$ . This, Proposition 7.1 and Corollary 7.2 complete the proof.  $\square$

**REMARK 7.4.** Concerning Corollary 7.3, we observe that, in view of (B.3),  $\mathbf{E}(g) := \mathbf{E}(g|\phi^{-1}(\mathcal{A})) = Pg$  a.e.  $[\mu]$  and thus  $C_\phi^*g = \mathbf{h}_\phi \cdot (\mathbf{E}(g) \circ \phi^{-1})$  for every  $g \in \mathcal{D}(C_\phi^*)$ , where  $\mathbf{E}(g) \circ \phi^{-1}$  is understood as in [11, Lemma 6.4].

## 8. Normality and quasinormality

It turns out that the characterizations of quasinormality and normality of unbounded composition operators take the same forms as those for bounded ones.

**PROPOSITION 8.1.** *If  $C_\phi$  is densely defined, then  $C_\phi$  is quasinormal if and only if  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$ .*

**PROOF.** Let  $C_\phi = U|C_\phi|$  be the polar decomposition of  $C_\phi$ . Suppose that  $C_\phi$  is quasinormal. Then by [48, Proposition 1],  $U|C_\phi| \subseteq |C_\phi|U$ . Let  $\{X_n\}_{n=1}^\infty$  be as in Corollary 4.6 (with  $m = 1$ ). Then, by (3.5),  $\{\chi_{X_n}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi)$ , which together with Proposition 7.1 implies that for every  $n \in \mathbb{N}$ ,

$$\chi_{X_n} \circ \phi = U|C_\phi|\chi_{X_n} = |C_\phi|U\chi_{X_n} = \left(\frac{\mathbf{h}_\phi}{\mathbf{h}_\phi \circ \phi}\right)^{1/2} \chi_{X_n} \circ \phi \quad \text{a.e. } [\mu].$$

Since  $X_n \nearrow X$  as  $n \rightarrow \infty$ , we conclude that  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$ .

For the converse, take  $f \in \mathcal{D}(|C_\phi|)$ . By (7.3) and  $\mathcal{D}(|C_\phi|) = \mathcal{D}(C_\phi)$ , we have

$$\mathbf{h}_\phi^{1/2}Uf = \left(\frac{\mathbf{h}_\phi}{\mathbf{h}_\phi \circ \phi}\right)^{1/2} f \circ \phi = f \circ \phi \in L^2(\mu).$$

Hence, by Proposition 7.1 (i),  $f \in \mathcal{D}(|C_\phi|U)$  and  $|C_\phi|Uf = C_\phi f = U|C_\phi|f$ . Therefore,  $U|C_\phi| \subseteq |C_\phi|U$ . Applying [48, Proposition 1] completes the proof.  $\square$

**PROPOSITION 8.2.** *If  $\overline{\mathcal{D}(C_\phi)} = L^2(\mu)$ , then the following are equivalent:*

- (i)  $C_\phi$  is normal,
- (ii)  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$  and  $\mathcal{N}(C_\phi^*) \subseteq \mathcal{N}(C_\phi)$ ,
- (iii)  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$  and  $\mathcal{N}(C_\phi^*) = \{0\}$ ,
- (iv)  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$  and for every  $\Delta \in \mathcal{A}$  there exists  $\Delta' \in \mathcal{A}$  such that  $\mu(\Delta \triangle \phi^{-1}(\Delta')) = 0$ .

Moreover, if  $C_\phi$  is normal, then  $\mathcal{N}(C_\phi) = \{0\}$  and  $\mathbf{h}_\phi > 0$  a.e.  $[\mu]$ .

**PROOF.** (i) $\Rightarrow$ (iii) Since normal operators are always quasinormal, we infer from Proposition 8.1 that  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$ . Clearly,  $\mathcal{N}(C_\phi) = \mathcal{N}(C_\phi^*)$ . That  $\mathcal{N}(C_\phi^*) = \{0\}$  follows from Corollary 6.6.

(iii) $\Rightarrow$ (ii) Evident.

(ii) $\Rightarrow$ (i) This is a direct consequence of Proposition 8.1 and Theorem 2.2.

(iii) $\Leftrightarrow$ (iv) Since  $\mathcal{N}(C_\phi^*) = \{0\}$  if and only if  $\mathcal{R}(C_\phi)$  is dense in  $L^2(\mu)$ , it suffices to apply Corollary 7.2, Lemma B.2 and (B.1).

The "moreover" part follows from the above and Proposition 6.5.  $\square$

### 9. Formal normality

In this section we show that formally normal composition operators are automatically normal. We begin by proving a result which is of measure-theoretical nature. We refer the reader to Appendix B for the definition and basic properties of  $E(\cdot|\phi^{-1}(\mathcal{A}))$ . For brevity, we write  $E(\cdot) = E(\cdot|\phi^{-1}(\mathcal{A}))$ .

LEMMA 9.1. *If  $\phi$  is a nonsingular transformation of  $X$ , then the following two conditions are equivalent for every  $n \in \mathbb{N}$ :*

- (i)  $h_{\phi^{n+1}} = h_{\phi^n} \cdot h_\phi$  a.e.  $[\mu]$ ,
- (ii)  $E(h_{\phi^n}) = h_{\phi^n} \circ \phi$  a.e.  $[\mu|_{\phi^{-1}(\mathcal{A})}]$ .

PROOF. (i) $\Rightarrow$ (ii) Note that

$$\begin{aligned} \int_{\phi^{-1}(\Delta)} E(h_{\phi^n}) d\mu &= \int_{\phi^{-1}(\Delta)} h_{\phi^n} d\mu = \mu((\phi^{-n}(\phi^{-1}(\Delta))) \\ &= \mu((\phi^{-(n+1)}(\Delta))) = \int_{\Delta} h_{\phi^{n+1}} d\mu = \int_{\Delta} h_{\phi^n} \cdot h_\phi d\mu \\ &= \int_X (\chi_{\Delta} \circ \phi)(h_{\phi^n} \circ \phi) d\mu = \int_{\phi^{-1}(\Delta)} h_{\phi^n} \circ \phi d\mu, \quad \Delta \in \mathcal{A}, \end{aligned}$$

which, by the uniqueness assertion in the Radon-Nikodym theorem, implies (ii).  $\square$

Arguing as above, we can prove the reverse implication.  $\square$

The next two lemmas are key ingredients of the proof of Theorem 9.4.

LEMMA 9.2. *Suppose that  $C_\phi^2$  is densely defined. Then the following two conditions are equivalent:*

- (i)  $\mathcal{D}(C_\phi^2) \subseteq \mathcal{D}(C_\phi^* C_\phi)$  and  $\|C_\phi^2 f\| = \|C_\phi^* C_\phi f\|$  for all  $f \in \mathcal{D}(C_\phi^2)$ ,
- (ii)  $h_{\phi^2} = h_\phi^2$  a.e.  $[\mu]$ .

Moreover, if  $C_\phi$  is formally normal, then (i) holds.

PROOF. (i) $\Rightarrow$ (ii) Take  $f \in \mathcal{D}(C_\phi^2)$ . Then, by Proposition 7.1(i), we have

$$\int_X |f|^2 h_\phi^2 d\mu = \|M_{h_\phi} f\|^2 = \|C_\phi^* C_\phi f\|^2 = \|C_{\phi^2} f\|^2 = \int_X |f|^2 h_{\phi^2} d\mu. \quad (9.1)$$

Let  $\{X_n\}_{n=1}^\infty$  be as in Corollary 4.6 (with  $m = 2$ ). Then  $\{\chi_{X_n}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi^2)$  and

$$\int_{\Delta} h_\phi^2 d\mu \stackrel{(9.1)}{=} \int_{\Delta} h_{\phi^2} d\mu < \infty, \quad \Delta \in \mathcal{A}, \Delta \subseteq X_n, n \in \mathbb{N},$$

which implies that  $h_\phi^2 = h_{\phi^2}$  a.e.  $[\mu]$  on  $X_n$  for every  $n \in \mathbb{N}$ . Hence (ii) holds.

(ii) $\Rightarrow$ (i) Take  $f \in \mathcal{D}(C_\phi^2)$ . Then, by (3.6),  $\int_X |f h_\phi|^2 d\mu = \int_X |f|^2 h_{\phi^2} d\mu < \infty$ , which means that  $f \in \mathcal{D}(M_{h_\phi}) = \mathcal{D}(C_\phi^* C_\phi)$ . Arguing as in (9.1), we obtain (i).  $\square$

The "moreover" part is obvious.  $\square$

LEMMA 9.3. *If  $\phi$  is nonsingular transformation of  $X$ , then the following conditions are equivalent:*

- (i)  $C_\phi$  is normal,
- (ii)  $C_\phi$  is formally normal and  $\overline{\mathcal{D}(C_\phi^2)} = L^2(\mu)$ .



PROOF. (i) $\Rightarrow$ (ii) Evident (since powers of normal operators are normal, cf. [3]).  
(ii) $\Rightarrow$ (i) First we will show that

$$\mathcal{D}(C_\phi) \cap \overline{\mathcal{R}(C_\phi)} = \mathcal{D}(C_\phi^*) \cap \overline{\mathcal{R}(C_\phi)}. \quad (9.2)$$

Indeed, if  $g \in \mathcal{D}(C_\phi^*) \cap \overline{\mathcal{R}(C_\phi)}$ , then by (7.2) there exists  $f \in L^2(\mathbf{h}_\phi d\mu)$  such that  $g = f \circ \phi$  a.e.  $[\mu]$ . It follows from Corollary 7.3 that  $\mathbf{h}_\phi f = \mathbf{h}_\phi V^{-1}g \in L^2(\mu)$ . This and Lemma 9.2 imply that

$$\int_X |g \circ \phi|^2 d\mu = \int_X |f \circ \phi^2|^2 d\mu = \int_X |f|^2 h_{\phi^2} d\mu = \int_X |f h_\phi|^2 d\mu < \infty,$$

which means that  $g \in \mathcal{D}(C_\phi)$ . This yields (9.2).

Let  $P$  be the orthogonal projection of  $L^2(\mu)$  onto  $\overline{\mathcal{R}(C_\phi)}$ . We will prove that

$$P\mathcal{D}(C_\phi) \subseteq \mathcal{D}(C_\phi). \quad (9.3)$$

Indeed, take  $f \in \mathcal{D}(C_\phi)$ . Since  $(I - P)f \in \mathcal{N}(C_\phi^*)$  and  $\mathcal{D}(C_\phi) \subseteq \mathcal{D}(C_\phi^*)$ , we get  $Pf \in \mathcal{D}(C_\phi^*) \cap \overline{\mathcal{R}(C_\phi)}$ . Hence by (9.2),  $Pf \in \mathcal{D}(C_\phi)$ , which proves (9.3).

It follows from (9.3) and Corollary 6.4 that

$$\mathcal{D}(C_\phi) \subseteq (\mathcal{D}(C_\phi) \cap \mathcal{N}(C_\phi^*)) \oplus (\mathcal{D}(C_\phi) \cap \overline{\mathcal{R}(C_\phi)}) = \mathcal{D}(C_\phi) \cap \overline{\mathcal{R}(C_\phi)},$$

which together with  $\overline{\mathcal{D}(C_\phi)} = L^2(\mu)$  imply that  $\overline{\mathcal{R}(C_\phi)} = L^2(\mu)$ . Therefore, by (9.2),  $\mathcal{D}(C_\phi) = \mathcal{D}(C_\phi^*)$ , which completes the proof.  $\square$

As is shown below, the assumption  $\overline{\mathcal{D}(C_\phi^2)} = L^2(\mu)$  in Lemma 9.3 can be dropped without spoiling its conclusion.

**THEOREM 9.4.** *Let  $\phi$  be a nonsingular transformation of  $X$ . Then  $C_\phi$  is normal if and only if  $C_\phi$  is formally normal.*

PROOF. It suffices to prove the “if” part. Suppose  $C_\phi$  is formally normal. Let  $\{X_n\}_{n=1}^\infty \subseteq \mathcal{A}$  be as in Corollary 4.6 (with  $m = 1$ ). Take  $\Delta \in \mathcal{A}$ . Since  $\{\chi_{X_n \cap \Delta}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi)$ , we get (see also Remark 7.4)

$$\begin{aligned} \int_{X_n \cap \Delta} \mathbf{h}_\phi d\mu &\stackrel{(3.4)}{=} \|C_\phi(\chi_{X_n \cap \Delta})\|^2 = \|C_\phi^*(\chi_{X_n \cap \Delta})\|^2 \\ &\stackrel{(7.8)}{=} \int_X \mathbf{h}_\phi^2 \cdot |V^{-1}E(\chi_{X_n \cap \Delta})|^2 d\mu \\ &= \int_X (\mathbf{h}_\phi \circ \phi)(E(\chi_{X_n \cap \Delta}))^2 d\mu, \quad n \in \mathbb{N}. \end{aligned}$$

Using (B.6) and Lebesgue’s monotone convergence theorem, we obtain

$$\int_\Delta \mathbf{h}_\phi d\mu = \int_X (\mathbf{h}_\phi \circ \phi)(E(\chi_\Delta))^2 d\mu, \quad \Delta \in \mathcal{A},$$

which yields

$$\int_{\phi^{-1}(\Delta)} \mathbf{h}_\phi d\mu = \int_{\phi^{-1}(\Delta)} \mathbf{h}_\phi \circ \phi d\mu, \quad \Delta \in \mathcal{A}.$$

This in turn implies that  $E(\mathbf{h}_\phi) = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu|_{\phi^{-1}(\mathcal{A})}]$ . By Lemma 9.1,  $\mathbf{h}_{\phi^2} = \mathbf{h}_\phi^2$  a.e.  $[\mu]$ . Since  $\mathbf{h}_\phi < \infty$  a.e.  $[\mu]$ , we see that  $\mathbf{h}_\phi + \mathbf{h}_{\phi^2} < \infty$  a.e.  $[\mu]$ . Using Corollary 4.5 (ii), we get  $\overline{\mathcal{D}(C_\phi^2)} = L^2(\mu)$ . Applying Lemma 9.3 completes the proof.  $\square$

REMARK 9.5. Using an unpublished result from [52] (based on a model for unbounded quasinormal operators), we can also prove Theorem 9.4 as follows. Suppose  $C_\phi$  is formally normal. Then, by the polarization formula, we have

$$\begin{aligned} \int_X f \bar{g} h_\phi d\mu &= \langle C_\phi f, C_\phi g \rangle = \langle C_\phi^* f, C_\phi^* g \rangle \\ &\stackrel{(7.8)}{=} \int_X h_\phi^2 (V^{-1} E(f)) (\overline{V^{-1} E(g)}) d\mu \\ &= \int_X (h_\phi \circ \phi) E(f) \overline{E(g)} d\mu, \quad f, g \in \mathcal{D}(C_\phi). \end{aligned} \quad (9.4)$$

By Propositions 3.2 and 6.1, and Corollary 6.3, we can assume that  $0 < h_\phi(x) < \infty$  for all  $x \in X$ . Let  $\{X_n\}_{n=1}^\infty \subseteq \mathcal{A}$  be as in Corollary 4.6 (with  $m = 1$ ). Set  $Y_n = \{x \in X_n : h_\phi(x) \geq 1/n\}$  for  $n \in \mathbb{N}$ . Clearly,  $Y_n \nearrow X$  as  $n \rightarrow \infty$ . Take  $\Delta \in \mathcal{A}$ . Since  $\{\chi_{Y_n}\}_{n=1}^\infty, \{h_\phi^{-1} \cdot \chi_{Y_n \cap \Delta}\}_{n=1}^\infty \subseteq \mathcal{D}(C_\phi)$ , we can substitute  $f = h_\phi^{-1} \cdot \chi_{Y_n \cap \Delta}$  and  $g = \chi_{Y_n}$  into (9.4). What we get is

$$\begin{aligned} \mu(Y_n \cap \Delta) &= \int_X (h_\phi \circ \phi) E(h_\phi^{-1} \cdot \chi_{Y_n \cap \Delta}) E(\chi_{Y_n}) d\mu \\ &\stackrel{(B.5)}{=} \int_{Y_n \cap \Delta} \frac{h_\phi \circ \phi}{h_\phi} E(\chi_{Y_n}) d\mu, \quad n \in \mathbb{N}. \end{aligned}$$

Using (B.6) and Lebesgue's monotone convergence theorem, we obtain

$$\mu(\Delta) = \int_\Delta \frac{h_\phi \circ \phi}{h_\phi} d\mu, \quad \Delta \in \mathcal{A},$$

which implies that  $h_\phi \circ \phi = h_\phi$  a.e.  $[\mu]$ . By Proposition 8.1,  $C_\phi$  is quasinormal. Since quasinormal formally normal operators are normal (cf. [52]), the proof is complete.

## 10. Generating Stieltjes moment sequences

We begin by proving two lemmas which are main tools in the proof of Theorem 10.4 below.

LEMMA 10.1. *Suppose  $\phi$  is a nonsingular transformation of  $X$  and  $\{\mathcal{E}_n\}_{n=1}^\infty$  is a sequence of subsets of  $L^2(\mu)$  satisfying the following three conditions:*

- (i)  $\mathcal{E}_n$  fulfils (A.5),  $\mathcal{E}_n \subseteq \mathcal{D}(C_\phi^n)$  and  $\overline{\mathcal{E}_n} = L^2(\mu)$  for all  $n \in \mathbb{N}$ ,
- (ii)  $[\|C_\phi^{i+j} f\|^2]_{i,j=0}^n \geq 0$  for all  $f \in \mathcal{E}_{2n}$  and  $n \in \mathbb{N}$ ,
- (iii)  $[\|C_\phi^{i+j+1} f\|^2]_{i,j=0}^n \geq 0$  for all  $f \in \mathcal{E}_{2n+1}$  and  $n \in \mathbb{N}$ .

*Then the following three assertions hold:*

- (a)  $\{h_{\phi^n}(x)\}_{n=0}^\infty$  is a Stieltjes moment sequence for  $\mu$ -a.e.  $x \in X$ ,
- (b)  $C_\phi^n = C_{\phi^n}$  for every  $n \in \mathbb{N}$ ,
- (c)  $\mathcal{D}^\infty(C_\phi)$  is a core for  $C_\phi^n$  for every  $n \in \mathbb{Z}_+$ .

PROOF. (a) By (i) and Corollary 4.5 (ii), there is no loss of generality in assuming that  $0 \leq h_{\phi^n}(x) < \infty$  for all  $x \in X$  and  $n \in \mathbb{Z}_+$ . Using (3.4), we obtain

$$\int_X \left| \sum_{i,j=0}^n \alpha_i \bar{\alpha}_j h_{\phi^{i+j}} \right| |f|^2 d\mu < \infty, \quad f \in \mathcal{D}(C_\phi^{2n}), \{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}, n \in \mathbb{Z}_+. \quad (10.1)$$

If  $\{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}$ , then by (i) and (ii) we have

$$0 \leq \sum_{i,j=0}^n \|C_\phi^{i+j} f\|^2 \alpha_i \bar{\alpha}_j = \int_X \left( \sum_{i,j=0}^n \alpha_i \bar{\alpha}_j h_{\phi^{i+j}} \right) |f|^2 d\mu, \quad f \in \mathcal{E}_{2n}, n \in \mathbb{N}.$$

Combining (i), (10.1) and Corollary A.6 (with  $\mathcal{E} = \mathcal{E}_{2n}$ ), we see that

$$\sum_{i,j=0}^n \alpha_i \bar{\alpha}_j h_{\phi^{i+j}} \geq 0 \text{ a.e. } [\mu] \text{ for all } n \in \mathbb{N} \text{ and } \{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}.$$

Let  $Q$  be a countable dense subset of  $\mathbb{C}$ . Then there exists a set  $\Delta_0 \in \mathcal{A}$  such that  $\mu(X \setminus \Delta_0) = 0$  and  $\sum_{i,j=0}^n \alpha_i \bar{\alpha}_j h_{\phi^{i+j}}(x) \geq 0$  for all  $n \in \mathbb{N}$ ,  $\{\alpha_i\}_{i=0}^n \subseteq Q$  and  $x \in \Delta_0$ . As  $Q$  is dense in  $\mathbb{C}$ , we conclude that  $[h_{\phi^{i+j}}(x)]_{i,j=0}^n \geq 0$  for all  $n \in \mathbb{N}$  and  $x \in \Delta_0$ . Using (iii) and applying a similar reasoning as above, we infer that there exists a set  $\Delta_1 \in \mathcal{A}$  such that  $\mu(X \setminus \Delta_1) = 0$  and  $[h_{\phi^{i+j+1}}(x)]_{i,j=0}^n \geq 0$  for all  $n \in \mathbb{N}$  and  $x \in \Delta_1$ . Employing (2.5) yields (a).

(b) By (a), there exists  $\Delta \in \mathcal{A}$  such that  $\mu(X \setminus \Delta) = 0$ ,  $h_{\phi^0}(x) = 1$  and  $\{h_{\phi^n}(x)\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $x \in \Delta$ . Hence, for every  $x \in \Delta$  there exists a Borel probability measure  $\mu_x$  on  $\mathbb{R}_+$  such that  $h_{\phi^n}(x) = \int_{\mathbb{R}_+} s^n d\mu_x(s)$  for all  $n \in \mathbb{Z}_+$ . This yields

$$\begin{aligned} \left( \sum_{j=0}^n h_{\phi^j} \right)(x) &= \int_{\mathbb{R}_+} \left( \sum_{j=0}^n s^j \right) d\mu_x(s) \\ &= \int_{[0,1)} \left( \sum_{j=0}^n s^j \right) d\mu_x(s) + \int_{[1,\infty)} \left( \sum_{j=0}^n s^j \right) d\mu_x(s) \\ &\leq (n+1) \int_{[0,1)} 1 d\mu_x(s) + (n+1) \int_{[1,\infty)} s^n d\mu_x(s) \\ &\leq (n+1)(1 + h_{\phi^n})(x), \quad x \in \Delta, n \in \mathbb{N}. \end{aligned}$$

Hence the domains of  $C_\phi^n$  and  $C_{\phi^n}$  coincide for all  $n \in \mathbb{N}$ . By (3.3), this gives (b).

(c) Apply (i) and Theorem 4.7. This completes the proof.  $\square$

LEMMA 10.2. *Suppose that  $\phi$  is a nonsingular transformation of  $X$  satisfying the following two conditions:*

- (i)  $\mathcal{D}(C_\phi^n)$  is dense in  $L^2(\mu)$  for every  $n \in \mathbb{N}$ ,
- (ii)  $[\|C_\phi^{i+j} f\|^2]_{i,j=0}^n \geq 0$  for all  $f \in \mathcal{D}(C_\phi^{2n})$  and  $n \in \mathbb{N}$ .

*Then the assertions (a), (b) and (c) of Lemma 10.1 hold.*

PROOF. Set  $\mathcal{E}_n = \mathcal{D}(C_\phi^n)$  for  $n \in \mathbb{N}$ . According to (3.6), each  $\mathcal{E}_n$  satisfies (A.5). Substituting  $C_\phi f$  for  $f$  in (ii) implies that the hypothesis (iii) of Lemma 10.1 is satisfied. Applying Lemma 10.1 completes the proof.  $\square$

COROLLARY 10.3. *If  $C_\phi$  is subnormal and  $\overline{\mathcal{D}(C_\phi^n)} = L^2(\mu)$  for all  $n \in \mathbb{N}$ , then the assertions (a), (b) and (c) of Lemma 10.1 hold.*

PROOF. Apply Proposition 2.4 and Lemma 10.2.  $\square$

The following theorem completely characterizes composition operators that generate Stieltjes moment sequences. It should be compared with Lambert's characterizations of bounded subnormal composition operators (cf. [31]).

THEOREM 10.4. *If  $\phi$  is a nonsingular transformation of  $X$ , then the following conditions are equivalent:*

- (i)  $C_\phi$  generates Stieltjes moment sequences,
- (ii)  $\{\mathbf{h}_{\phi^n}(x)\}_{n=0}^\infty$  is a Stieltjes moment sequence for  $\mu$ -a.e.  $x \in X$ ,
- (iii)  $\mathcal{D}(C_\phi^k) = L^2(\mu)$  for all  $k \in \mathbb{N}$ , and  $\{\mu(\phi^{-n}(\Delta))\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $\Delta \in \mathcal{A}$  such that  $\mu(\phi^{-k}(\Delta)) < \infty$  for all  $k \in \mathbb{Z}_+$ ,
- (iv)  $\mathbf{h}_{\phi^n} < \infty$  a.e.  $[\mu]$  for all  $n \in \mathbb{N}$  and  $L(p) \geq 0$  a.e.  $[\mu]$  whenever  $p(t) \geq 0$  for all  $t \in \mathbb{R}_+$ , where  $L: \mathbb{C}[t] \rightarrow \mathcal{M}$  is a linear mapping determined by<sup>6</sup>

$$L(t^n) = \mathbf{h}_{\phi^n}, \quad n \in \mathbb{Z}_+;$$

here  $\mathbb{C}[t]$  is the set of all complex polynomials in one real variable  $t$  and  $\mathcal{M}$  is the set of all  $\mathcal{A}$ -measurable complex functions on  $X$ .

Moreover, if (i) holds, then  $C_\phi^n = C_{\phi^n}$  and  $\mathcal{D}^\infty(C_\phi)$  is a core for  $C_\phi^n$  for all  $n \in \mathbb{Z}_+$ .

PROOF. (i) $\Rightarrow$ (ii) Set  $\mathcal{E}_n = \mathcal{D}^\infty(C_\phi)$  for  $n \in \mathbb{N}$ . By (2.5), (3.6) and Lemma 10.1, we see that the condition (ii) and the “moreover” part hold.

(ii) $\Rightarrow$ (i) Take  $f \in \mathcal{D}^\infty(C_\phi)$ ,  $n \in \mathbb{Z}_+$  and  $\{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}$ . Then, by (2.5), we have

$$\sum_{i,j=0}^n \alpha_i \bar{\alpha}_j \|C_\phi^{i+j} f\|^2 \stackrel{(3.4)}{=} \int_X \left( \sum_{i,j=0}^n \alpha_i \bar{\alpha}_j \mathbf{h}_{\phi^{i+j}}(x) \right) |f(x)|^2 d\mu(x) \geq 0.$$

Applying the above to  $C_\phi f$  in place of  $f$ , we deduce that the sequences  $\{\|C_\phi^k f\|^2\}_{k=0}^\infty$  and  $\{\|C_\phi^{k+1} f\|^2\}_{k=0}^\infty$  are positive definite. Therefore, by (2.5),  $\{\|C_\phi^k f\|^2\}_{k=0}^\infty$  is a Stieltjes moment sequence. It follows from Corollary 4.5(ii) and Theorem 4.7 that  $\mathcal{D}^\infty(C_\phi)$  is dense in  $L^2(\mu)$ .

(i) $\Rightarrow$ (iii) Evident (because  $\chi_\Delta \in \mathcal{D}^\infty(C_\phi)$  for every  $\Delta$  as in (iii)).

(iii) $\Rightarrow$ (i) By Theorem 4.7 the set  $\mathcal{D}^\infty(C_\phi)$  is dense in  $L^2(\mu)$ . Consider a simple  $\mathcal{A}$ -measurable function  $u = \sum_{i=1}^k \alpha_i \chi_{\Delta_i}$ , where  $\{\alpha_i\}_{i=1}^k$  are positive real numbers and  $\{\Delta_i\}_{i=1}^k$  are pairwise disjoint sets in  $\mathcal{A}$ . Suppose that  $u$  is in  $\mathcal{D}^\infty(C_\phi)$ . Then, by the measure transport theorem,  $\{\chi_{\Delta_i}\}_{i=1}^k \subseteq \mathcal{D}^\infty(C_\phi)$  and

$$\|C_\phi^n u\|^2 = \sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Delta_i \cap \Delta_j} \mathbf{h}_{\phi^n} d\mu = \sum_{i=1}^k \alpha_i^2 \int_{\Delta_i} \mathbf{h}_{\phi^n} d\mu \stackrel{(3.4)}{=} \sum_{i=1}^k \alpha_i^2 \mu(\phi^{-n}(\Delta_i))$$

for all  $n \in \mathbb{Z}_+$ . Hence, by (iii), we have

$$\{\|C_\phi^n u\|^2\}_{n=0}^\infty \text{ is a Stieltjes moment sequence for every simple nonnegative } \mathcal{A}\text{-measurable function } u \in \mathcal{D}^\infty(C_\phi). \quad (10.2)$$

Now take  $f \in \mathcal{D}^\infty(C_\phi)$ . Then there exists a sequence  $\{u_n\}_{n=1}^\infty$  of simple  $\mathcal{A}$ -measurable functions  $u_n: X \rightarrow \mathbb{R}_+$  such that  $u_n(x) \leq u_{n+1}(x) \leq |f(x)|$  and  $\lim_{k \rightarrow \infty} u_k(x) = |f(x)|$  for all  $n \in \mathbb{N}$  and  $x \in X$ . This implies that  $\{u_n\}_{n=1}^\infty \subseteq \mathcal{D}^\infty(C_\phi)$  and, by Lebesgue's monotone convergence theorem,

$$\|C_\phi^n f\|^2 = \int_X |f|^2 \mathbf{h}_{\phi^n} d\mu = \lim_{k \rightarrow \infty} \int_X u_k^2 \mathbf{h}_{\phi^n} d\mu = \lim_{k \rightarrow \infty} \|C_\phi^n u_k\|^2, \quad n \in \mathbb{Z}_+.$$

Since the class of Stieltjes moment sequences is closed under the operation of taking pointwise limits (cf. (2.5)), we infer from (10.2) that  $\{\|C_\phi^n f\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence.

<sup>6</sup> To make the definition of  $L$  correct we have to modify  $\mathbf{h}_{\phi^n}$  so that  $0 \leq \mathbf{h}_{\phi^n}(x) < \infty$  for all  $x \in X$  and  $n \in \mathbb{Z}_+$ .

(ii) $\Rightarrow$ (iv) If  $p \in \mathbb{C}[t]$  is such that  $p(t) \geq 0$  for all  $t \in \mathbb{R}_+$ , then there exist  $q_1, q_2 \in \mathbb{C}[t]$  such that  $p(t) = t|q_1(t)|^2 + |q_2(t)|^2$  for all  $t \in \mathbb{R}$  (see [36, Problem 45, p. 78]). This fact combined with (2.5) implies that  $L(p) \geq 0$  a.e.  $[\mu]$ .

(iv) $\Rightarrow$ (ii) Let  $Q$  be a countable dense subset of  $\mathbb{C}$ . If  $q \in \mathbb{C}[t]$  is a polynomial with coefficients in  $Q$ , then the polynomials  $p_1 := |q|^2$  and  $p_2 := t|q|^2$  are nonnegative on  $\mathbb{R}_+$ . Hence  $L(p_i) \geq 0$  a.e.  $[\mu]$  for  $i = 1, 2$ . Since  $Q$  is countable, this implies that there exists  $\Delta \in \mathcal{A}$  such that  $\mu(X \setminus \Delta) = 0$ ,

$$0 \leq h_{\phi^n}(x) < \infty, \sum_{i,j=0}^n \alpha_i \bar{\alpha}_j h_{\phi^{i+j}}(x) \geq 0 \text{ and } \sum_{i,j=0}^n \alpha_i \bar{\alpha}_j h_{\phi^{i+j+1}}(x) \geq 0 \quad (10.3)$$

for all  $n \in \mathbb{Z}_+$ ,  $\{\alpha_i\}_{i=0}^n \subseteq Q$  and  $x \in \Delta$ . As  $Q$  is dense in  $\mathbb{C}$ , we see that (10.3) holds for all  $n \in \mathbb{Z}_+$ ,  $\{\alpha_i\}_{i=0}^n \subseteq \mathbb{C}$  and  $x \in \Delta$ . This and (2.5) complete the proof.  $\square$

**CONCLUSION 10.5.** We close the paper by pointing out that there exists a composition operator generating Stieltjes moment sequences which is not subnormal and even not hyponormal. Such an operator can be constructed on the basis of a weighted shift on a directed tree with one branching vertex (cf. [25, Section 4.3]). In view of Theorem 10.4, any composition operator  $C_\phi$  which generates Stieltjes moment sequences, in particular the aforementioned, satisfies the conditions (ii), (iii) and (iv) of this theorem as well as its “moreover” part (specifically,  $\mathcal{D}^\infty(C_\phi)$  is a core for  $C_\phi^n$  for every  $n \in \mathbb{Z}_+$ , which is considerably more than is required in Definition 2.3). Therefore, none of the Lambert characterizations of subnormality of bounded composition operators (cf. [31]) is valid in the unbounded case. It is worth mentioning that the above example is built over the discrete measure space. However, it can be immediately adapted to the context of measures which are equivalent to the Lebesgue measure on  $[0, \infty)$  by applying [24, Theorem 2.7].

## Appendix A.

Here we gather some useful properties of  $L^2$ -spaces. The first two lemmas seem to be folklore. For the reader’s convenience, we include their proofs.

**LEMMA A.1.** *Let  $(X, \mathcal{A}, \mu)$  be a measure space and let  $\rho_1, \rho_2$  be  $\mathcal{A}$ -measurable scalar functions on  $X$  such that  $0 < \rho_i < \infty$  a.e.  $[\mu]$  for  $i = 1, 2$ . Then  $L^2(\rho_1 d\mu) \cap L^2(\rho_2 d\mu)$  is dense<sup>7</sup> in  $L^2(\rho_i d\mu)$  for  $i = 1, 2$ .*

**PROOF.** Since  $L^2(\rho_1 d\mu) \cap L^2(\rho_2 d\mu) = L^2((\rho_1 + \rho_2) d\mu)$ , we can assume that  $0 < \rho_2(x) \leq \rho_1(x) < \infty$  for all  $x \in X$ . Take  $\Delta \in \mathcal{A}$  such that  $\chi_\Delta \in L^2(\rho_2 d\mu)$ . Set  $\Delta_n = \{x \in \Delta: \rho_1(x) \leq n \text{ and } \frac{1}{n} \leq \rho_2(x)\}$  for  $n \in \mathbb{N}$ . Note that  $\{\chi_{\Delta_n}\}_{n=1}^\infty \subseteq L^2(\rho_1 d\mu)$ . Since  $\Delta_n \nearrow \Delta$  as  $n \rightarrow \infty$ , we see that  $\{\chi_{\Delta_n}\}_{n=1}^\infty$  converges to  $\chi_\Delta$  in  $L^2(\rho_2 d\mu)$ . Applying [38, Theorem 3.13] completes the proof.  $\square$

Note that Lemma A.1 is no longer true if one of the density functions  $\rho_1$  and  $\rho_2$  takes the value  $\infty$  on a set of positive measure  $\mu$  (even if  $\rho_2 \leq \rho_1$ ). Employing Lemma A.1 and the Radon-Nikodym theorem, we get the following.

**COROLLARY A.2.** *Let  $(X, \mathcal{A}, \mu_1)$  and  $(X, \mathcal{A}, \mu_2)$  be  $\sigma$ -finite measure spaces. If the measures  $\mu_1$  and  $\mu_2$  are mutually absolutely continuous, then  $L^2(\mu_1) \cap L^2(\mu_2)$  is dense in  $L^2(\mu_i)$  for  $i = 1, 2$ .*

Corollary A.2 is no longer true if one of the measures  $\mu_1$  and  $\mu_2$  is not  $\sigma$ -finite.

<sup>7</sup> This makes sense because the measures  $\rho_1 d\mu$  and  $\rho_2 d\mu$  are mutually absolutely continuous.

LEMMA A.3. *Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $\rho_1, \rho_2$  be  $\mathcal{A}$ -measurable scalar functions on  $X$  such that  $0 < \rho_1 \leq \infty$  a.e.  $[\mu]$  and  $0 \leq \rho_2 \leq \infty$  a.e.  $[\mu]$ . Then the following two conditions are equivalent:*

- (i)  $\int_X |f|^2 \rho_2 d\mu < \infty$  for every  $\mathcal{A}$ -measurable function  $f: X \rightarrow \mathbb{C}$  such that  $\int_X |f|^2 \rho_1 d\mu < \infty$ ,
- (ii) *there exists  $c \in \mathbb{R}_+$  such that  $\rho_2 \leq c\rho_1$  a.e.  $[\mu]$ .*

PROOF. (i) $\Rightarrow$ (ii) Without loss of generality we can assume that  $\rho_1 < \infty$  a.e.  $[\mu]$ . We can also assume that  $\rho_2 < \infty$  a.e.  $[\mu]$  (indeed, otherwise, since  $\rho_1 < \infty$  a.e.  $[\mu]$  and  $\mu$  is  $\sigma$ -finite, there exist  $\Omega \in \mathcal{A}$  and  $k \in \mathbb{N}$  such that  $\rho_1(x) \leq k$  and  $\rho_2(x) = \infty$  for all  $x \in \Omega$ , and  $0 < \mu(\Omega) < \infty$ ; hence  $\int_\Omega \rho_1 d\mu < \infty$  and  $\int_\Omega \rho_2 d\mu = \infty$ , which is a contradiction). Finally, replacing  $\rho_2$  by  $\frac{\rho_2}{\rho_1}$  if necessary, we can assume that  $\rho_1(x) = 1$  for all  $x \in X$ . Now applying the Landau-Riesz summability theorem (cf. [5, Problem G, p. 398]), we obtain (ii). The implication (ii) $\Rightarrow$ (i) is obvious.  $\square$

COROLLARY A.4. *Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $\rho_1, \rho_2$  be  $\mathcal{A}$ -measurable scalar functions on  $X$  such that  $0 < \rho_i \leq \infty$  a.e.  $[\mu]$  for  $i = 1, 2$ . Then  $L^2(\rho_1 d\mu) \subseteq L^2(\rho_2 d\mu)$  if and only if there exists  $c \in \mathbb{R}_+$  such that  $\rho_2 \leq c\rho_1$  a.e.  $[\mu]$ .*

The implication (i) $\Rightarrow$ (ii) of Lemma A.3 is not true if we drop the assumption that  $\rho_1 > 0$  a.e.  $[\mu]$ . Corollary A.4 is no longer true if  $\mu$  is not  $\sigma$ -finite (e.g.,  $X = \mathbb{N}$ ,  $\mathcal{A} = 2^X$ ,  $\mu(\{1\}) = 1$ ,  $\mu(\{i\}) = \infty$  for  $i \geq 2$ ,  $\rho_1 \equiv 1$  and  $\rho_2(n) = n$  for  $n \in X$ ).

The following lemma generalizes [24, Lemma 2.1].

LEMMA A.5. *Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space,  $\mathcal{E}$  be a dense subset of  $L^2(\mu)$  and  $h: X \rightarrow \mathbb{C}$  be an  $\mathcal{A}$ -measurable function such that*

$$\int_\Delta |h||f|^2 d\mu < \infty \text{ and } \int_\Delta h|f|^2 d\mu \geq 0 \text{ for all } f \in \mathcal{E} \text{ and } \Delta \in \mathcal{A}_*, \quad (\text{A.1})$$

where  $\mathcal{A}_* = \{\Delta \in \mathcal{A} : \mu(\Delta) < \infty\}$ . Then  $h \geq 0$  a.e.  $[\mu]$ .

PROOF. Set  $\Xi_f = \{x \in X : |f(x)| > 0\}$  for  $f \in \mathcal{E}$ . First, we will show that

$$h(x) \geq 0 \text{ for } \mu\text{-a.e. } x \in \Xi_f \text{ and for every } f \in \mathcal{E}. \quad (\text{A.2})$$

Indeed, fix  $f \in \mathcal{E}$  and set  $\Xi_{f,k} = \{x \in X : |f(x)| \geq \frac{1}{k}\}$  for  $k \in \mathbb{N}$ . It follows from Chebyshev's inequality that  $\Xi_{f,k} \in \mathcal{A}_*$  for  $k \in \mathbb{N}$ . Applying (A.1), we deduce that

$$\int_{\Xi_{f,k}} |h||f|^2 d\mu < \infty \text{ and } \int_{\Xi_{f,k} \cap \Delta} h|f|^2 d\mu \geq 0 \text{ for all } \Delta \in \mathcal{A} \text{ and } k \in \mathbb{N}.$$

This implies that  $h \geq 0$  a.e.  $[\mu]$  on  $\Xi_{f,k}$  for every  $k \in \mathbb{N}$ . Since  $\Xi_{f,k} \nearrow \Xi_f$  as  $k \rightarrow \infty$ , we conclude that  $h \geq 0$  a.e.  $[\mu]$  on  $\Xi_f$ .

Set  $\Sigma = \{x \in X : h(x) \geq 0\}$ . Suppose that, contrary to our claim,  $\mu(X \setminus \Sigma) > 0$ . As  $\mu$  is  $\sigma$ -finite, there exists a set  $\Omega \in \mathcal{A}$  such that  $\Omega \subseteq X \setminus \Sigma$  and  $0 < \mu(\Omega) < \infty$ . This means that  $\chi_\Omega \in L^2(\mu)$ . Since  $\mathcal{E}$  is dense in  $L^2(\mu)$ , there exists a sequence  $\{f_n\}_{n=1}^\infty \subseteq \mathcal{E}$  which converges to  $\chi_\Omega$  in  $L^2(\mu)$ . Passing to a subsequence if necessary, we can assume that the sequence  $\{f_n\}_{n=1}^\infty$  converges a.e.  $[\mu]$  to  $\chi_\Omega$ , and thus

$$\lim_{n \rightarrow \infty} f_n(x) = \chi_\Omega(x) = 1 \text{ for } \mu\text{-a.e. } x \in \Omega. \quad (\text{A.3})$$

It follows from (A.2) that  $\mu(\Omega \cap \Xi_f) = 0$  for every  $f \in \mathcal{E}$ . Applying this property to  $f = f_n$  ( $n \in \mathbb{N}$ ), we see that  $\mu\left(\Omega \cap \bigcup_{n=1}^\infty \Xi_{f_n}\right) = 0$ , which means that

$$f_n(x) = 0 \text{ for all } n \in \mathbb{N} \text{ and for } \mu\text{-a.e. } x \in \Omega. \quad (\text{A.4})$$

Combining (A.3) with (A.4), we conclude that  $\mu(\Omega) = 0$ , a contradiction.  $\square$

Applying Lemma A.5 to  $h$  and  $-h$ , we see that this lemma remains valid if “ $\geq$ ” is replaced by “ $=$ ”.

**COROLLARY A.6.** *Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $\mathcal{E}$  be a dense subset of  $L^2(\mu)$  such that*

$$f\chi_\Delta \in \mathcal{E} \text{ for all } f \in \mathcal{E} \text{ and } \Delta \in \mathcal{A}_*. \quad (\text{A.5})$$

*If  $h: X \rightarrow \mathbb{C}$  is an  $\mathcal{A}$ -measurable function such that  $\int_X |h||f|^2 d\mu < \infty$  and  $\int_X h|f|^2 d\mu \geq 0$  for all  $f \in \mathcal{E}$ , then  $h \geq 0$  a.e.  $[\mu]$ .*

### Appendix B.

In this appendix, we describe (mostly without proofs) some results from measure theory which play an important role in our analysis of composition operators. Let  $(X, \mathcal{A}, \mu)$  be a fixed measure space and let  $\mathcal{B} \subseteq \mathcal{A}$  be a  $\sigma$ -algebra. We say that  $\mathcal{B}$  is *relatively  $\mu$ -complete* if  $\mathcal{A}_0 \subseteq \mathcal{B}$ , where  $\mathcal{A}_0 = \{\Delta \in \mathcal{A} : \mu(\Delta) = 0\}$  (cf. [23]). It is easily seen that the smallest relatively  $\mu$ -complete  $\sigma$ -algebra containing  $\mathcal{B}$ , denoted by  $\mathcal{B}^\mu$ , coincides with the  $\sigma$ -algebra generated by  $\mathcal{B} \cup \mathcal{A}_0$ , and that

$$\mathcal{B}^\mu = \{\Delta \in \mathcal{A} \mid \exists \Delta' \in \mathcal{B} : \mu(\Delta \triangle \Delta') = 0\}. \quad (\text{B.1})$$

The  $\mathcal{B}^\mu$ -measurable functions are described below (cf. [38, Lemma 1, p. 169]).

**LEMMA B.1.** *A function  $f: X \rightarrow \mathbb{C}$  is  $\mathcal{B}^\mu$ -measurable if and only if there exists a  $\mathcal{B}$ -measurable function  $g: X \rightarrow \mathbb{C}$  such that  $f = g$  a.e.  $[\mu]$ .*

By the above lemma  $L^2(\mu|_{\mathcal{B}})$  is a subset of  $L^2(\mu)$  if and only if  $\mathcal{B} = \mathcal{B}^\mu$ . The question of when  $L^2(\mu|_{\mathcal{B}}) = L^2(\mu)$  has a simple answer ( $\sigma$ -finiteness is essential!).

**LEMMA B.2.** *If  $\mu$  is  $\sigma$ -finite and  $\mathcal{B}$  is relatively  $\mu$ -complete, then  $L^2(\mu|_{\mathcal{B}}) = L^2(\mu)$  if and only if  $\mathcal{B} = \mathcal{A}$ .*

**PROOF.** Suppose that  $L^2(\mu|_{\mathcal{B}}) = L^2(\mu)$  and  $\Delta \in \mathcal{A} \setminus \mathcal{B}$ . Since  $\mu$  is  $\sigma$ -finite, we may assume that  $\mu(\Delta) < \infty$ . Then  $\chi_\Delta \in L^2(\mu) \setminus L^2(\mu|_{\mathcal{B}})$ , a contradiction.  $\square$

Given a transformation  $\phi$  of  $X$ , we set  $\phi^{-1}(\mathcal{A}) = \{\phi^{-1}(\Delta) : \Delta \in \mathcal{A}\}$ .

**LEMMA B.3.** *Suppose that  $\phi: X \rightarrow X$  is an  $\mathcal{A}$ -measurable transformation and  $f: X \rightarrow \mathbb{C}$  is an arbitrary function. Then  $f$  is  $(\phi^{-1}(\mathcal{A}))^\mu$ -measurable if and only if there exists an  $\mathcal{A}$ -measurable function  $u: X \rightarrow \mathbb{C}$  such that  $f = u \circ \phi$  a.e.  $[\mu]$ .*

**PROOF.** Applying the following well-known fact

$$\begin{aligned} &\text{a function } g: X \rightarrow \mathbb{C} \text{ is } \phi^{-1}(\mathcal{A})\text{-measurable if and only if there} \\ &\text{exists an } \mathcal{A}\text{-measurable function } u: X \rightarrow \mathbb{C} \text{ such that } g = u \circ \phi, \end{aligned} \quad (\text{B.2})$$

and Lemma B.1 completes the proof.  $\square$

Let  $P_{\mathcal{B}}$  be the orthogonal projection of  $L^2(\mu)$  onto its closed subspace  $L^2(\mu|_{\mathcal{B}^\mu})$ . Set  $\mathcal{B}_* = \{\Delta \in \mathcal{B} : \mu(\Delta) < \infty\}$ . It follows from Lemma B.1 that

$$\begin{aligned} &\text{for every } f \in L^2(\mu) \text{ there exists a unique (up to sets of measure zero)} \\ &\mathcal{B}\text{-measurable function } E(f|\mathcal{B}): X \rightarrow \mathbb{C} \text{ such that } P_{\mathcal{B}}f = E(f|\mathcal{B}) \text{ a.e. } [\mu]. \end{aligned} \quad (\text{B.3})$$

This and the fact that  $\langle \chi_\Delta, f \rangle = \langle \chi_\Delta, P_{\mathcal{B}}f \rangle$  for all  $f \in L^2(\mu)$  and  $\Delta \in \mathcal{B}_*$  yield

$$\int_\Delta f d\mu = \int_\Delta E(f|\mathcal{B}) d\mu, \quad f \in L^2(\mu), \Delta \in \mathcal{B}_*. \quad (\text{B.4})$$



Now suppose that  $\mu|_{\mathcal{B}}$  is  $\sigma$ -finite. It follows from (B.4) that  $E(f|\mathcal{B}) \geq 0$  a.e.  $[\mu]$  whenever  $f \geq 0$  a.e.  $[\mu]$ . By applying the standard approximation procedure, we see that for every  $\mathcal{A}$ -measurable function  $f: X \rightarrow [0, \infty]$  there exists a unique (up to sets of measure zero)  $\mathcal{B}$ -measurable function  $E(f|\mathcal{B}): X \rightarrow [0, \infty]$  such that the equality in (B.4) holds for every  $\Delta \in \mathcal{B}$ . Thus for every  $\mathcal{A}$ -measurable function  $f: X \rightarrow [0, \infty]$  and for every  $\mathcal{B}$ -measurable function  $g: X \rightarrow [0, \infty]$  we have

$$\int_X gf \, d\mu = \int_X gE(f|\mathcal{B}) \, d\mu. \quad (\text{B.5})$$

We call  $E(f|\mathcal{B})$  the *conditional expectation* of  $f$  with respect to  $\mathcal{B}$  (cf. [37]). Clearly,

$$\text{if } 0 \leq f_n \nearrow f \text{ are } \mathcal{A}\text{-measurable, then } E(f_n|\mathcal{B}) \nearrow E(f|\mathcal{B}), \quad (\text{B.6})$$

where  $g_n \nearrow g$  means that for  $\mu$ -a.e.  $x \in X$ , the sequence  $\{g_n(x)\}_{n=1}^{\infty}$  is monotonically increasing and convergent to  $g(x)$ .

Concluding Appendix B, we note that if  $\mu$  is  $\sigma$ -finite and  $\phi: X \rightarrow X$  is a nonsingular transformation such that  $h_{\phi} < \infty$  a.e.  $[\mu]$  (equivalently,  $C_{\phi}$  is densely defined), then the measure  $\mu|_{\phi^{-1}(\mathcal{A})}$  is  $\sigma$ -finite (cf. Proposition 3.2). Thus we may consider the conditional expectation  $E(\cdot|\phi^{-1}(\mathcal{A}))$  with respect to  $\phi^{-1}(\mathcal{A})$ .

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